Day 4 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Conservation of momentum (CIE)

Particles A (8 kg) and B (2 kg) move in the same direction on a smooth horizontal line at 3 m s^{-1} and 8 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):
$8\times3 + 2\times8 = 10\,v$, so $v = 4$
KE before = $\frac{1}{2}\times8\times3^2 + \frac{1}{2}\times2\times8^2 = 100$
KE after = $\frac{1}{2}\times10\times4^2 = 80$
Loss = $100 - 80$

Answer: (a) 4 m s^{-1} (b) 20 J

2 Stratified sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A college has 216 students: 96 Arts, 48 Science and 72 Humanities.
A stratified sample of size 18 is taken.
Find the number of students sampled from each group.

Sampling fraction = $\frac{18}{216} = \frac{1}{12}$

Arts: $96 \times \frac{1}{12} = 8$

Science: $48 \times \frac{1}{12} = 4$

Humanities: $72 \times \frac{1}{12} = 6$

Answer: Arts 8, Science 4, Humanities 6

3 Equilibrium on a slope: range of force (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 8 kg particle rests on a rough plane inclined at α, $\tan\alpha = \frac{4}{3}$, $\mu = 0.25$. A force of P N acts up a line of greatest slope and the particle stays at rest. Find the range of values of P. Take $g = 9.8$.

$\sin\alpha = 0.8$, $\cos\alpha = 0.6$, $R = 8g\cos\alpha = 47.04$ N
Limiting friction = $\mu R = 11.76$ N
P smallest: friction acts up the slope: $P + \mu R = 8g\sin\alpha$, $P = 62.72 - 11.76$
P greatest: friction acts down the slope: $P = 62.72 + 11.76$

Answer: $50.96 \text{ N} \leq P \leq 74.48$ N

4 Arithmetic sequences: finding a and d (All boards)

The 4th term of an arithmetic sequence is 27 and the 12th term is 67.
Find the first term and common difference, and the sum of the first 25 terms.

$u_{12} - u_4 = 8d = 67 - 27$ so $d = 5$
$a + 3d = 27$ so $a = 12$
$S_{25} = \frac{25}{2}(2\times12 + 24\times5)$

Answer: $a = 12$, $d = 5$, $S_{25} = 1800$

5 Differentiating sec, cosec, tan and cot (All boards)

$y = \sec(2x)$.
Find $\frac{dy}{dx}$ and its exact value at $x = \frac{\pi}{8}$.

$\frac{dy}{dx} = 2\sec(2x)\tan(2x)$

At $x = \frac{\pi}{8}$ the angle $2x = \frac{\pi}{4}$, so the gradient is $2\sqrt{2}$

Answer: $\frac{dy}{dx} = 2\sec(2x)\tan(2x)$; $2\sqrt{2}$

6 Three-set Venn diagram counts (All boards)

Of 37 students, 17 study Maths (M), 17 Physics (P) and 18 Chemistry (C).
9 study M and P, 10 study M and C, 9 study P and C, and 5 study all three.
One student is chosen at random. Given that a student studies exactly two subjects, find the probability that they study Physics.

Venn regions: M only 3, P only 4, C only 4,
MP only 4, MC only 5, PC only 4, all three 5, outside all 8
Exactly two subjects: 4 + 5 + 4 = 13
Of these, Physics: 4 + 4 = 8

Probability = $\frac{8}{13}$

Answer: $\mathbf{\frac{8}{13}}$

7 Implicit differentiation: horizontal tangents (All boards)

A curve has equation $4x^2 - 2xy + 2y^2 = 112$.
Find the coordinates of the points where the tangent is parallel to the x-axis.

$\frac{dy}{dx} = -\frac{8x - 2y}{-2x + 4y}$
Horizontal tangent: $8x - 2y = 0$, so $y = 4x$
Substitute: $28x^2 = 112$, so $x^2 = 4$ and $x = \pm 2$
Points: $(2, 8)$ and $(-2, -8)$
Answer: (2, 8) and (−2, −8)

8 Binomial expansion: finding n and k (All boards)

In the expansion of $(1 + kx)^n$, where n is a positive integer, the coefficient of x is 24 and the coefficient of x^2 is 252.
Find n and k.

$nk = 24$ and $\frac{n(n-1)}{2}k^2 = 252$

$(n-1)k = \frac{2 \times 252}{24} = 21$ (dividing the second equation by the first)

$nk - (n-1)k = k = 24 - 21 = 3$

$n = \frac{24}{3} = 8$

Answer: n = 8, k = 3

9 Proof by contradiction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove by contradiction that there are no integers m and n such that $10m + 15n = 31$.

Assume $10m + 15n = 31$ for integers m, n.
Then $5(2m + 3n) = 31$, so 5 divides 31.
But 31 is not a multiple of 5. Contradiction.
Answer: no such integers exist

10 Integration by parts (All boards)

Use integration by parts to find the exact value of

$\int_0^{\frac{\pi}{6}} x \sin 3x \, dx$.

$u = x$, $\frac{dv}{dx} = \sin 3x$: $\int x \sin 3x \, dx = \left[-\frac{x \cos 3x}{3}\right] + \int \frac{\cos 3x}{3} \, dx$

$= \left[-\frac{x \cos 3x}{3} + \frac{\sin 3x}{9}\right]$

$= \frac{1}{9}$

Answer: $\mathbf{\frac{1}{9}}$