Day 365 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

The volume of a spherical balloon increases at a constant rate of 120 cm^3s^{-1}.
Find the rates of increase of its radius and of its surface area when r = 2 cm.

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$. $\frac{dr}{dt} = \frac{dV}{dt \div dV/dr} = \frac{120}{4\pi r^2}$

r = 2: $\frac{dr}{dt} = \frac{15}{2\pi}$ cm s^{-1}

$A = 4\pi r^2$, so $\frac{dA}{dr} = 8\pi r$ and $\frac{dA}{dt} = 8\pi \times 2 \times \frac{15}{2\pi} = 120$ cm^2s^{-1}

Answer: $\mathbf{\frac{dr}{dt} = \frac{15}{2\pi}}$ **cm/s,** $\mathbf{\frac{dA}{dt} = 120\ cm^2/s}$

2 Arithmetic modelling (All boards)

A charity raises £400 in its first year and in each later year raises £50 more than in the previous year.
(a) How much does it raise in year 20? (b) In which year does the total raised first exceed £6000?

(a) $u_{20} = 400 + 19 \times 50 = 1350$

(b) $S_m = \frac{m}{2}(800 + (m - 1) \times 50) > 6000$

$S_9 = 5400 < 6000$ and $S_{10} = 6250 > 6000$

Answer: (a) £1350 (b) year 10

3 Constant speed against friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A crate of mass 20 kg is pulled at constant speed along rough horizontal ground, $\mu = 0.3$, by a force of P N at angle θ above the horizontal, $\tan\theta = \frac{12}{5}$. Find P and the normal reaction. Take g = 9.8. Give answers to 3 s.f. if not exact.

Constant speed, so the resultant force is zero. $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$

Vertical: $R = 20\ g - P\sin\theta$
Horizontal: $P\cos\theta = \mu R = 0.3(20\ g - P\sin\theta)$

$P(\frac{5}{13} + \frac{18}{65}) = 0.3 \times 20\ g$

P = 88.9, R = 114

Answer: P = 88.9 N, R = 114 N

4 Line and curve intersection (diagram) (All boards)

The diagram shows the curve $y = x^2 + x - 7$ and the line $y = -x - 4$, meeting at A and B.
(a) Find the coordinates of A and B.
(b) Find the coordinates of the midpoint of AB.

$x^2 + x - 7 = -x - 4$: $x^2 + 2x - 3 = 0$
$(x + 3)(x - 1) = 0$, $x = -3$ or 1
A(−3, −1), B(1, −5)
Midpoint: average the coordinates

Answer: (a) A(−3, −1), B(1, −5) (b) (−1, −3)

5 Pulley, string goes slack (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (9 kg) and B (5 kg) hang on a light string over a smooth pulley, A being 2 m above the ground. Released from rest; A lands without rebounding. Find (a) the tension while both move, (b) the greatest height of B above its start. Take g = 9.8. Give answers to 3 s.f. if not exact.

A: $9\ g - T = 9\ a$; B: $T - 5\ g = 5\ a$
$a = 4\ g \div 14 = 2.8$, so $T = 5(g + a) = 63$
When A lands, B has risen 2 m with $v^2 = 2 \times 2.8 \times 2 = 11.2$

B then moves freely under gravity: extra height = $v^2 \div (2 \times 9.8) = \frac{4}{7}$

Total = $2 + \frac{4}{7}$

Answer: (a) 63 N (b) 2.57 m

6 Selections with combinations (OCR, MEI, Eduqas, CIE)

A group of 4 boys and 3 girls. A team of 3 is chosen at random.
Find the probability that all the team are of the same sex.

Total teams = C(7, 3) = 35
Same sex: C(4, 3) + C(3, 3) = 4 + 1 = 5

$P = \frac{5}{35}$

$= \frac{1}{7}$

Answer: $\frac{1}{7}$

7 Reducing a relationship to a straight line (graph) (All boards)

The diagram shows the graph of ln y against x for $y = A e^{kx}$.
Find the exact values of A and k.

ln y = ln A + kx, a straight line with gradient k and intercept ln A

Intercept ln A = 2, so $A = e^2$

Gradient $k = \frac{6 - 2}{2 - 0} = 2$

Answer: $\mathbf{A = e^2, k = 2}$

8 Surds: equation with surd coefficients (AQA, Edexcel, OCR, MEI, Eduqas)

Solve $x(\sqrt{2} - 1) = -4 - 3\sqrt{2}$, writing x in the form $p + q\sqrt{2}$.

$x = \frac{-4 - 3\sqrt{2}}{\sqrt{2} - 1}$

Multiply top and bottom by $\sqrt{2} + 1$.

Numerator: $(-4 - 3\sqrt{2})(\sqrt{2} + 1) = -10 - 7\sqrt{2}$

Denominator: $2 - 1 = 1$

Answer: $\mathbf{x = -10 - 7\sqrt{2}}$

9 Double and triple angle values (All boards)

Given that $\cos\theta = \frac{2}{5}$, find the exact value of $\cos 3\theta$.

$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \cos\theta = 4\cos^3\theta - 3\cos\theta$

$= 4 \times \left(\frac{2}{5}\right)^3 - 3 \times \frac{2}{5} = -\frac{118}{125}$

Answer: $\mathbf{-\frac{118}{125}}$

10 Binomial in two stages (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Eggs are defective with probability 0.2, independently. A tray of 6 eggs is rejected if it has at least 3 defective.
From 11 such trays, find the probability that exactly 3 are rejected (3 s.f.).

$Y \sim B(6, 0.2)$. $P(\text{rejected}) = P(Y \geq 3) = 1 - 0.901 = 0.0989$

$W \sim B(11, 0.0989)$ is the number rejected

$P(W = 3) = C(11, 3)(0.0989)^3(1 - 0.0989)^8 = 0.0694$

Answer: 0.0694