Day 357 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Speed-time graphs (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the speed-time graph for a train journey of 960 m. Find (a) V, (b) the deceleration in the final stage. Give answers to 3 s.f. if not exact.

(a) Distance = area of trapezium = ½(80 + 40) × V
960 = 60 V, so V = 16
(b) Deceleration = V ÷ 15 = 16 ÷ 15
Answer: (a) V = 16 (b) 1.07 m s^{-2}

2 Parametric differentiation: tangent and normal (All boards)

A curve has parametric equations $x = t^2 + 1$, $y = t^3 - t$.

Find $\frac{dy}{dx}$ in terms of t, and the equation of the normal at the point where $t = \frac{1}{2}$, in the form ax + by + c = 0 (integers).

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = \frac{3t^2 - 1}{2t}$

$t = \frac{1}{2}$: point $(\frac{5}{4}, -\frac{3}{8})$, gradient $-\frac{1}{4}$

Normal gradient = $-1 \div (-\frac{1}{4}) = 4$, so normal: $32x - 8y - 43 = 0$

Answer: $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$; $32x - 8y - 43 = 0$

3 Poisson probabilities (CIE)

Cars pass a junction at a mean rate of 5 per day, as a Poisson process.
Find the probability that in one such period there are
(a) exactly 6 (b) at least 6 (c) fewer than 6 (3 s.f.).

X ~ Po(5)
(a) $P(X = 6) = e^{-5}\, 5^6/720! = 0.146$
(b) $1 - P(X \leq 5) = 1 - 0.616 = 0.384$
(c) $P(X \leq 5) = 0.616$
Answer: (a) 0.146 (b) 0.384 (c) 0.616

4 Conservation of momentum (CIE)

Particles A (2 kg) and B (4 kg) move in the same direction on a smooth horizontal line at 5 m s^{-1} and 5 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):
2×5 + 4×5 = 6 v, so v = 5
KE before = ½×2×5^2 + ½×4×5^2 = 75
KE after = ½×6×5^2 = 75
Loss = 75 − 75
Answer: (a) 5 m s^{-1} (b) 0 J

5 Collinear points and ratios (3D) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A, B and C have position vectors $\begin{pmatrix}4\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}7\\-1\\-4\end{pmatrix}$ and $\begin{pmatrix}-2\\k\\-4\end{pmatrix}$

relative to O, where k is a constant. Given that A, B and C are collinear, find k and the ratio AB : BC.

$\overrightarrow{AB} = \begin{pmatrix}3\\2\\0\end{pmatrix}$, and $\overrightarrow{AC} = \begin{pmatrix}-6\\k+3\\0\end{pmatrix}$

$\overrightarrow{AC} = \lambda \overrightarrow{AB}$: component 2 gives $-4 = 2\lambda$, so $\lambda = -2$
Hence $k = -3 + (-2) \times 2 = -7$
$\overrightarrow{AC} = -2\,\overrightarrow{AB}$, so $\overrightarrow{BC} = -3\,\overrightarrow{AB}$: AB : BC = 1 : 3
Answer: k = −7, AB : BC = 1 : 3

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6 Inverse of a rational function (All boards)

$f(x) = \frac{5x-1}{4x+5}$, $x \in \mathbb{R}$, $x \neq -\frac{5}{4}$.

Find $f^{-1}(x)$ and state its domain.

$y = \frac{5x-1}{4x+5}$: $y(4x+5) = 5x-1$

$4xy + 5y = 5x - 1$

$x(4y-5) = -5y - 1$

$x = \frac{-5y-1}{4y-5}$, then swap x and y

Answer: $f^{-1}(x) = \frac{-5x-1}{4x-5}$, $x \neq \frac{5}{4}$

7 R-form and solving a sin x + b cos x = c (All boards)

(a) Express $16\cos\theta - 12\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give α to 1 d.p.

(b) Solve $16\cos\theta - 12\sin\theta = 19$ for $0° \leq \theta < 360°$, giving answers to 1 d.p.

$R^2 = 16^2 + 12^2 = 400$, so $R = 20$; $\tan\alpha = \frac{12}{16}$ gives $\alpha = 36.9°$

$20\cos(\theta + 36.9°) = 19$, so $\cos(\theta + 36.9°) = \frac{19}{20}$

Principal value 18.2°, then use the symmetry of the cosine graph

$\theta = 304.9°, 341.3°$

Answer: (a) $R = 20$, $\alpha = 36.9°$ (b) $\theta = 304.9°, 341.3°$

8 Circle through two points with centre on a line (AQA, Edexcel, OCR, MEI, Eduqas)

A circle passes through A(4, −9) and B(−13, 8).

Its centre lies on the line $y = 3x + 6$.

Find the equation of the circle.

Centre $(h, 3h + 6)$ is equidistant from A and B.

$(h-4)^2 + (3h+15)^2 = (h+13)^2 + (3h-2)^2$

Solving gives $h = -1$, so centre $(-1, 3)$

$r^2 = (5)^2 + (-12)^2 = 169$

Answer: $(x+1)^2 + (y-3)^2 = 169$

9 Range of a composite function (All boards)

$f(x) = x^2 - 2x + 3$, $x \in \mathbb{R}$, and $g(x) = \frac{3}{x+2}$, $x > -2$.

(a) Find the range of f.

(b) Show that gf is defined and find its range.

$f(x) = (x-1)^2 + 2 \geq 2$

So $f(x) \geq 2 > -2$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{3}{f(x)+2}$, with $f(x) \geq 2$, so $f(x) + 2 \geq 4$

Largest value is $\frac{3}{4} = \frac{3}{4}$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 2$ (b) $0 < gf(x) \leq \frac{3}{4}$

10 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The life of a battery is $N(63, 4^2)$. A battery is classed as 'special' if its life is more than 70.

10 batterys are chosen at random. Find the probability that exactly 2 are special (3 s.f.).

$P(\text{special}) = P(X > 70) = P(Z > 1.75) = 0.0401$

$Y \sim B(10, 0.0401)$

$P(Y = 2) = C(10, 2)(0.0401)^2(0.960)^8 = 0.0521$

Answer: 0.0521