Day 344 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Ships on collision courses (vectors) (AQA, Edexcel, OCR, MEI, Eduqas)

At 12:00 ship P has position $(8\mathbf{i} + 13\mathbf{j})$ km and velocity $(-3\mathbf{i} - \mathbf{j})$ km h^{-1}; ship Q has position $(15\mathbf{j})$ km and velocity $(\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Show that the ships collide and find the time and position vector of the collision.

At time t hours: $\mathbf{r}_P = (8\mathbf{i} + 13\mathbf{j}) + t(-3\mathbf{i} - \mathbf{j})$, $\mathbf{r}_Q = (15\mathbf{j}) + t(\mathbf{i} - 2\mathbf{j})$
Equal when: **i**: $8 + -3t = 0 + 1t$, so $t = 2$
j: $13 + 2\times(-1) = 11$
Check with Q: **i**: $0 + 2\times1 = 2$, **j**: $15 + 2\times(-2) = 11$
Both components agree at $t = 2$, so they collide at $(2\mathbf{i} + 11\mathbf{j})$ km
Answer: t = 2 hours (at 14:00); r = (2i + 11j) km

2 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has $P(X = x) = k(x + 1)$ for $x = 0, 1, 2, 3$.

(a) Show that $k = \frac{1}{10}$. (b) Find E(X) and Var(X).

(c) Find $P(X \geq 1)$.

(a) $\Sigma P(X = x) = 1$: $1 + 2 + 3 + 4 = 10$, so $10k = 1$

(b) $E(X) = 1 \times \frac{1}{5} + 2 \times \frac{3}{10} + 3 \times \frac{2}{5} = 2$

$E(X^2) = 5$, so $Var(X) = 5 - (2)^2 = 1$

(c) $P(X \geq 1) = \frac{1}{5} + \frac{3}{10} + \frac{2}{5} = \frac{9}{10}$

Answer: (b) E(X) = 2, Var(X) = 1 (c) $\frac{9}{10}$

3 Transformations from a graph (All boards)

The diagram shows the curve y = f(x), with turning points A and B.
Find the coordinates of the turning points of
(a) y = f(x + 3) (b) y = −f(x)

(a) y = f(x + 3): A → (−3, 7), B → (−1, −1)
(b) y = −f(x): A → (0, −7), B → (2, 1)
Answer: (a) (−3, 7), (−1, −1); (b) (0, −7), (2, 1)

4 Speed-time graphs (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the speed-time graph for a train journey of 1625 m. Find (a) V, (b) the deceleration in the final stage. Give answers to 3 s.f. if not exact.

(a) Distance = area of trapezium = ½(80 + 45) × V
1625 = 62.5 V, so V = 26
(b) Deceleration = V ÷ 15 = 26 ÷ 15
Answer: (a) V = 26 (b) 1.73 m s^{-2}

5 R-form and solving a sin x + b cos x = c (All boards)

(a) Express $24 \sin x + 7 \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give α to 1 d.p.
(b) Solve $24 \sin x + 7 \cos x = 20$ for $0° \leq x < 360°$, giving answers to 1 d.p.

$R^2 = 24^2 + 7^2 = 625$, so R = 25; $\tan \alpha = \frac{7}{24}$ gives $\alpha = 16.3°$

$25 \sin(x + 16.3°) = 20$, so $\sin(x + 16.3°) = \frac{4}{5}$

Principal value 53.1°, then use the symmetry of the sine graph
x = 36.9°, 110.6°
Answer: (a) R = 25, α = 16.3° (b) x = 36.9°, 110.6°

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6 Binomial expansion: terms with negative powers of x (All boards)

Find the coefficient of x^{10} in the expansion of

$(2x^2 - \frac{1}{x})^8$.

General term: ${}^8C_r\,(2x^2)^{8-r}\,(-x^{-1})^r$

Power of x: $2(8 - r) - 1r = 10$

$16 - 3r = 10$ so $r = 2$

Coefficient = ${}^8C_2 \times 2^6 \times (-1)^2 = 1792$

Answer: 1792

7 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially h = 36, and after 2 hours h = 25.

Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\frac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\,dh = \int -k\,dt$, so $2\sqrt{h} = -kt + C$

$t = 0$: $C = 2 \times 6 = 12$, so $2\sqrt{h} = 12 - kt$

$t = 2$, $h = 25$: $10 = 12 - 2k$, so $k = 1$

Empty when $h = 0$: $kt = 12$, so $t = \frac{12}{1} = 12$ hours

Answer: 12 hours

8 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas)

A circle touches both coordinate axes and passes through the point $(-6, 3)$. Its centre lies in the second quadrant.

Find the equations of the two possible circles.

Centre $(-r, r)$, radius $r > 0$.

Through $(-6, 3)$: $(-6 + r)^2 + (3 - r)^2 = r^2$

$r^2 - 18r + 45 = 0$

$(r - 3)(r - 15) = 0$, so $r = 3$ or 15

Answer: $(x + 3)^2 + (y - 3)^2 = 9$ or $(x + 15)^2 + (y - 15)^2 = 225$

9 Double angle equations (All boards)

Solve $\cos 2x + \sin x = 0$

for $-180° \le x \le 180°$.

$\cos 2x \equiv 1 - 2\sin^2 x$: $2\sin^2 x - \sin x - 1 = 0$

Solve the quadratic: $\sin x = 1$ or $\sin x = -\frac{1}{2}$

$\sin x = 1$: $x = 90°$

$\sin x = -\frac{1}{2}$: $x = -150°, -30°$

Answer: $x = -150°, -30°, 90°$

10 Two-tailed binomial test critical region (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(30, p)$. H_0: $p = 0.3$ is tested against H_1: $p \neq 0.3$ at the 10% level.

(a) Find the critical region.

(b) Find the actual significance level of the test (3 s.f.).

(a) Each tail ≤ 0.05. $P(X \le 4) = 0.0302$, $P(X \le 5) = 0.0766$

$P(X \ge 14) = 0.0401$, $P(X \ge 13) = 0.0845$

Critical region: $X \le 4$ or $X \ge 14$

(b) Actual level = $0.0302 + 0.0401 = 0.0702$

Answer: (a) $X \le 4$ or $X \ge 14$ (b) 0.0702