Day 340 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has P(X = x) = k(7 − x) for x = 2, 3, 4, 5, 6.

(a) Show that k = $\frac{1}{15}$. (b) Find E(X) and Var(X).

(c) Find P(X ≥ 4).

(a) Σ P(X = x) = 1: 5 + 4 + 3 + 2 + 1 = 15, so 15k = 1

(b) E(X) = $2 \times \frac{1}{3} + 3 \times \frac{4}{15} + 4 \times \frac{1}{5} + 5 \times \frac{2}{15} + 6 \times \frac{1}{15} = \frac{10}{3}$

$E(X^2) = \frac{38}{3}$, so $Var(X) = \frac{38}{3} - (\frac{10}{3})^2 = \frac{14}{9}$

(c) $P(X \geq 4) = \frac{1}{5} + \frac{2}{15} + \frac{1}{15} = \frac{2}{5}$

Answer: (b) E(X) = $\frac{10}{3}$, Var(X) = $\frac{14}{9}$ (c) $\frac{2}{5}$

2 Fixed point iteration (All boards)

The equation $e^x = 4x + 5$ has a root α. Use the iteration $x_{n+1} = \ln(4x + 5)$ with $x_0 = 3$ to find x_1, x_2 and x_3 to 4 d.p., and hence the value of α to 2 d.p. (continue until the 2 d.p. value is settled).

$x_1 = \ln(4 \times 3 + 5) = 2.8332$

$x_2 = 2.7932$, $x_3 = 2.7833$

Continuing, the values settle at 2.7801, so α = 2.78

Answer: x_1 = 2.8332, x_2 = 2.7932, x_3 = 2.7833; α = 2.78

3 Newton's second law with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

A particle of mass 3 kg is acted on by forces $F_1 = (-\mathbf{i} + 12\mathbf{j})$ N and F_2. Its acceleration is $(-3\mathbf{i} - \mathbf{j})$ m s^{-2}. Find (a) F_2, (b) the angle between F_2 and $\mathbf{i}$, to 1 d.p.

F = ma: $F_1 + F_2 = 3(-3\mathbf{i} - \mathbf{j}) = -9\mathbf{i} - 3\mathbf{j}$

(a) $F_2 = (-9\mathbf{i} - 3\mathbf{j}) - (-\mathbf{i} + 12\mathbf{j}) = -8\mathbf{i} - 15\mathbf{j}$

(b) $|F_2| = \sqrt{(-8^2 + -15^2)} = 17$

cos θ = −8 ÷ 17, so θ = 118.1°

Answer: (a) $F_2 = (-8\mathbf{i} - 15\mathbf{j})$ N (b) 118.1°

4 Greatest and least values using R-form (All boards)

$g(\theta) = \frac{1}{18 + 15 \sin\theta + 8 \cos\theta}$

(a) Write 15 sin θ + 8 cos θ in the form R sin(θ + α), with α acute.

(b) Find the greatest and least values of g(θ) and the smallest positive θ at which the greatest value occurs (1 d.p.).

$R = \sqrt{15^2 + 8^2} = 17$, $\alpha = \tan^{-1}\frac{8}{15} = 28.1°$

Denominator = 18 + 17 sin(θ + 28.1°), ranging from 1 to 35

Greatest g = $\frac{1}{1}$ (denominator least), least g = $\frac{1}{35}$

Least denominator when sin(θ + 28.1°) = −1: θ + 28.1° = 270°

θ = 241.9°

Answer: R = 17, α = 28.1°; greatest $\frac{1}{1}$, least $\frac{1}{35}$; θ = 241.9°

5 Inverse of a quadratic function (All boards)

$f(x) = x^2 + 4x + 10$, $x \geq -2$.

Find $f^{-1}(x)$ and state its domain.

$f(x) = (x + 2)^2 + 6$

Let y = f(x): $(x + 2)^2 = y - 6$

$x \geq -2$, so take the positive root: $x = -2 + \sqrt{y - 6}$

Answer: $f^{-1}(x) = -2 + \sqrt{x - 6}$, $x \geq 6$

6 Smooth incline held by a string (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 4 kg particle rests on a smooth plane inclined at α, $\tan\alpha = \frac{5}{12}$, held by a string in the vertical plane through a line of greatest slope, at angle θ above the slope, $\tan\theta = \frac{5}{12}$. Find the tension and the normal reaction. Take g = 9.8. Give answers to 3 s.f. if not exact.

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$

Along the plane: $T\cos\theta = 4g\sin\alpha$, so $T = 16.3$

Perpendicular: $R + T\sin\theta = 4g\cos\alpha$

$R = 4g\cos\alpha - T\sin\theta = 36.2 - 16.3\times0.385$

Answer: T = 16.3 N, R = 29.9 N

7 Circle touching an axis (diagram) (AQA, Edexcel, OCR, MEI, Eduqas)

The diagram shows a circle that touches the x-axis at A(−2, 0) and passes through B(−6, 2).
Find the equation of the circle.

Touches the x-axis at A, so the centre is (−2, r) and the radius is r.

B on the circle: $(-4)^2 + (2 - r)^2 = r^2$

$16 + 4 - 4r = 0$ so $r = 5$

Answer: $(x + 2)^2 + (y - 5)^2 = 25$

8 Two-tailed binomial test critical region (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(26, p)$. H_0: $p = 0.3$ is tested against H_1: $p \neq 0.3$ at the 5% level.

(a) Find the critical region.

(b) Find the actual significance level of the test (3 s.f.).

(a) Each tail ≤ 0.025. $P(X \leq 2) = 0.00674$, $P(X \leq 3) = 0.0260$

$P(X \geq 14) = 0.00944$, $P(X \geq 13) = 0.0255$

Critical region: $X \leq 2$ or $X \geq 14$

(b) Actual level = 0.00674 + 0.00944 = 0.0162

Answer: (a) $X \leq 2$ or $X \geq 14$ (b) 0.0162

9 Small angle approximations: solving an equation (AQA, Edexcel, OCR, MEI, Eduqas)

θ is small and measured in radians. Use small angle approximations to show that

$3\sin 2\theta + \cos 3\theta = 2$

can be written as $a\theta^2 + b\theta + c = 0$ with integers a, b, c. Hence find the approximate solution with $0 < \theta < 0.8$, to 3 s.f.

$\sin 2\theta \approx 2\theta$, $\cos 3\theta \approx 1 - \frac{9}{2}\theta^2$

$3 \times 2\theta + 1(1 - \frac{9}{2}\theta^2) = 2$

Multiply by 2 and rearrange: $9\theta^2 - 12\theta + 2 = 0$

$\theta = \frac{12 \pm \sqrt{72}}{18}$, so $\theta = 0.195$

Answer: a = 9, b = −12, c = 2; $\theta \approx 0.195$

Day 340 **A Level Maths: A Tier 10-a-day: Model Solutions**

10 Binomial expansion: approximating a square root (All boards)

(a) Expand $(1 + x)^{\frac{1}{2}}$ in ascending powers of x up to x^2.

(b) Use x = 0.02 to estimate $\sqrt{102}$ to 4 decimal places.

$(1 + y)^{\frac{1}{2}} = 1 + \frac{1}{2}y - \frac{1}{8}y^2$, y = 1x

(a) $1 + \frac{1}{2}x - \frac{1}{8}x^2$

$x = \frac{1}{50}$: $102 = 10^2(1 + \frac{1}{50})$ so $\sqrt{102} = 10 \times (1 + \frac{1}{50})^{\frac{1}{2}}$

$\approx 10 \times (1 + \frac{1}{100} - \frac{1}{20000}) = 10.099500$

Answer: (a) $1 + \frac{1}{2}x - \frac{1}{8}x^2$ (b) 10.0995