Day 33 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Chain rule and gradients (All boards)

$y = \sin^3(x)$. Find $\frac{dy}{dx}$ and its exact value at $x = \frac{\pi}{3}$.

Let $u = \sin(x)$: $y = u^3$, so $\frac{dy}{dx} = 3u^2 \times \frac{du}{dx}$

$\frac{du}{dx} = \cos(x)$, giving $\frac{dy}{dx} = 3\sin^2(x)\cos(x)$

At $x = \frac{\pi}{3}$: $\sin(x) = \frac{\sqrt{3}}{2}$ and $\cos(x) = \frac{1}{2}$, so gradient $= \frac{9}{8}$

Answer: $\frac{dy}{dx} = 3\sin^2(x)\cos(x)$; gradient $= \frac{9}{8}$

2 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = -x - 6$ and $g(x) = x^2 + 2$.
Solve $gf(x) = 18$.

$gf(x) = g(-x - 6) = (-x - 6)^2 + 2$
$(-x - 6)^2 + 2 = 18$
$-x - 6 = \pm 4$
Answer: $x = -10$ or $x = -2$

3 Lift and cable tension (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 500 kg carries a person of mass 75 kg. The cable tension is 6095 N and the lift is accelerating. Find (a) the magnitude and direction of the acceleration, (b) the force of the floor on the person. Take $g = 9.8$.

Whole system (mass 575 kg), taking upwards as positive: $T - 575g = 575a$
$6095 - 5635 = 575a$, so $a = 0.8$
(a) a is positive, so the acceleration is 0.8 m s^{-2} upwards
(b) Person: $R - 75g = 75a$, so $R = 75(9.8 + 0.8)$
Answer: (a) 0.8 m s^{-2} upwards (b) 795 N

4 Geometric distribution (CIE)

On each independent trial, the probability that a bulb is faulty is $\frac{1}{5}$.

X is the number of trials up to and including the first success.
(a) Find $P(X = 6)$ and $E(X)$.
(b) Find the least m with $P(X \leq m) > 0.9$.

(a) $P(X = 6) = (\frac{4}{5})^5 \times \frac{1}{5} = \frac{1024}{15625}$; $E(X) = \frac{1}{p} = 5$

(b) $P(X \leq m) = 1 - (\frac{4}{5})^m > 0.9$

$(\frac{4}{5})^m < 0.1$, so $m > 10.3$

Least integer $m = 11$

Answer: (a) $\frac{1024}{15625}$, $E(X) = 5$ (b) $m = 11$

5 Differentiating sec, cosec, tan and cot (All boards)

$y = x\,\text{cosec}(2x)$.

Find the exact gradient at $x = \frac{\pi}{8}$.

Product rule: $\frac{dy}{dx} = \text{cosec}(2x) + x(-2\text{cosec}(2x)\cot(2x))$

At $x = \frac{\pi}{8}$: $\text{cosec}(2x) = \sqrt{2}$ and the derivative term $= -2\sqrt{2}$

Gradient $= \sqrt{2} + \frac{\pi}{8} \times (-2\sqrt{2}) = \frac{4\sqrt{2} - \sqrt{2}\pi}{4}$

Answer: gradient $\frac{4\sqrt{2} - \sqrt{2}\pi}{4}$

6 Conditional probability and independence (All boards)

P(B) = 0.35, P(A) = 0.3 and P(A | B) = 0.25.
(a) Find P(A ∩ B) and P(A ∪ B).
(b) Find P(B | A).
(c) Determine whether A and B are independent.

(a) $P(A \cap B) = P(A \mid B)P(B) = 0.25 \times 0.35 = \frac{7}{80}$

$P(A \cup B) = 0.3 + 0.35 - \frac{7}{80} = \frac{9}{16}$

(b) $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{7}{80}}{0.3} = \frac{7}{24}$

(c) $P(A)P(B) = 0.3 \times 0.35 = 0.105$, which does not equal $P(A \cap B) = \frac{7}{80}$

So A and B are not independent

Answer: (a) $\frac{7}{80}$, $\frac{9}{16}$ (b) $\frac{7}{24}$ (c) not independent

7 Stationary points with trigonometric and exponential terms (All boards)

$y = 2x - 4\sin x$ for $0 \leq x \leq 2\pi$.
Find the exact coordinates of the stationary points and determine their nature.

$\frac{dy}{dx} = 2 - 4\cos x = 0$, so $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$ in the range

$\frac{d^2y}{dx^2} = 4\sin x$: sign at each point gives the nature

Answer: $\left(\frac{\pi}{3}, \frac{2\pi - 6\sqrt{3}}{3}\right)$ minimum; $\left(\frac{5\pi}{3}, \frac{6\sqrt{3} + 10\pi}{3}\right)$ maximum

8 Cubic inequalities (All boards)

Given that $(x - 2)$ is a factor of $f(x) = -x^3 - 2x^2 + 13x - 10$, solve $f(x) \leq 0$.

Factorise: $f(x) = -(x + 5)(x - 1)(x - 2)$
Roots at $x = -5, 1, 2$
Sketch the cubic (negative x^3 coefficient: positive, negative, positive, negative)
Read off where $f(x) \leq 0$.
Answer: $-5 \leq x \leq 1$ or $x \geq 2$

9 Periodic sequences (All boards)

A sequence is defined by $u_1 = \frac{3}{2}$ and $u_{n+1} = \frac{u_n + 1}{u_n - 1}$.

(a) Find u_2 and u_3.

(b) Find u_{100} and $\sum_{r=1}^{100} u_r$.

$u_2 = 5$, $u_3 = \frac{3}{2}$, $u_4 = 5$

$u_3 = u_1$, so the sequence repeats with period 2.
$100 = 2 \times 50 + 0$, so $u_{100} = u_2 = 5$

Sum of one period = $\frac{13}{2}$; 50 full periods plus 0 extra term(s)

Answer: (a) 5, $\frac{3}{2}$ (b) $u_{100} = 5$, sum = 325

10 Ladder in limiting equilibrium (AQA, Edexcel, OCR, MEI)

A uniform ladder AB (10 m, 15 kg) rests with A on rough ground, μ = 0.25, and B against a smooth wall, B being 8 m above the ground. A man of mass 75 kg climbs the ladder. How far along the ladder from A can he climb before it slips?

Foot of ladder is $\sqrt{(10^2 - 8^2)} = 6$ m from the wall

Vertical: $R = (15 + 75)g$. Limiting friction: $F = \mu R = 0.25(15 + 75)g$

Horizontal: $S = F$ (wall reaction S)

Moments about A (man at distance x along ladder, horizontal distance $x \times 6/10$):

$S \times 8 = 15g \times 3 + 75g \times (6x \div 10)$

$22.5 \times 8 = 45 + 45x$ (g cancels), so $x = 3$

Answer: 3 m