Day 328 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Histograms (All boards)

The histogram shows the masses of apples, m (in grams), of a sample.
(a) How many were in the class $40 \leq m < 60$?
(b) Estimate how many had $35 < m < 55$.

(a) Frequency = frequency density × width = 5 × 20 = 100
(b) 35 to 40: 3 × 5 = 15
40 to 55: 5 × 15 = 75
Total = 90
Answer: (a) 100 (b) 90

2 Newton's second law with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

A particle of mass 2 kg is acted on by forces $F_1 = (14\mathbf{i} - 11\mathbf{j})$ N and F_2. Its acceleration is $(\mathbf{i} - 3\mathbf{j})$ m s^{-2}. Find (a) F_2, (b) the angle between F_2 and $\mathbf{i}$, to 1 d.p.

$F = ma$: $F_1 + F_2 = 2(\mathbf{i} - 3\mathbf{j}) = 2\mathbf{i} - 6\mathbf{j}$
(a) $F_2 = (2\mathbf{i} - 6\mathbf{j}) - (14\mathbf{i} - 11\mathbf{j}) = -12\mathbf{i} + 5\mathbf{j}$
(b) $|F_2| = \sqrt{(-12^2 + 5^2)} = 13$
$\cos\theta = -12 \div 13$, so $\theta = 157.4°$
Answer: (a) $F_2 = (-12\mathbf{i} + 5\mathbf{j})$ N (b) 157.4°

3 Dividing a line in a ratio (2D position vectors) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B have position vectors $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. The point P lies on AB with AP : PB = 2 : 3.
Find the position vector of P and the exact distance OP.

$\text{OP} = \text{OA} + \frac{2}{5}\text{AB} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\text{OP} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$

Answer: $\begin{pmatrix} \mathbf{0} \\ \mathbf{-1} \end{pmatrix}$, OP = 1

4 Conservation of momentum (CIE)

Particles A (6 kg) and B (3 kg) move towards each other on a smooth horizontal line at 5 m s^{-1} and 4 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):
$6\times5 - 3\times4 = 9v$, so $v = 2$
KE before = $\frac{1}{2}\times6\times5^2 + \frac{1}{2}\times3\times4^2 = 99$
KE after = $\frac{1}{2}\times9\times2^2 = 18$
Loss = $99 - 18$
Answer: (a) 2 m s^{-1} (b) 81 J

5 Differentiating sec, cosec, tan and cot (All boards)

$y = \cot(2x)$.
Find $\frac{dy}{dx}$ and its exact value at $x = \frac{\pi}{6}$.

$\frac{dy}{dx} = -2\text{cosec}^2(2x)$

At $x = \frac{\pi}{6}$ the angle $2x = \frac{\pi}{3}$, so the gradient is $-\frac{8}{3}$

Answer: $\frac{dy}{dx} = -2\text{cosec}^2(2x)$; $-\frac{8}{3}$

6 Recurrence relations: unknown first term (All boards)

A sequence is defined by $u_1 = k$ and $u_{n+1} = 3u_n - 4$.

Given that $u_3 = -61$, find (a) the value of k, (b) $\sum_{r=1}^{4} u_r$.

$u_2 = 3k - 4$, $u_3 = 3(3k - 4) - 4$
$9k - 16 = -61$ so $k = -5$
Terms: $-5, -19, -61, -187$
Sum $= -272$
Answer: (a) $k = -5$ (b) -272

7 Binomial in two stages (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Parts are defective with probability 0.25, independently. A box of 9 parts is rejected if it has at least 4 defective. From 9 such boxes, find the probability that exactly 2 are rejected (3 s.f.).

$Y \sim B(9, 0.25)$. $P(\text{rejected}) = P(Y \geq 4) = 1 - 0.834 = 0.166$
$W \sim B(9, 0.166)$ is the number rejected
$P(W = 2) = C(9, 2)(0.166)^2(1 - 0.166)^7 = 0.278$
Answer: 0.278

8 Sum to infinity: two possible ratios (All boards)

The second term of a geometric series is 6 and its sum to infinity is 27.
Find the two possible values of the first term.

$ar = 6$ and $\frac{a}{(1 - r)} = 27$
$a = 27(1 - r)$, so $27(1 - r)\,r = 6$
$27r^2 - 27r + 6 = 0$: $r = \frac{1}{3}$ or $r = \frac{2}{3}$
$a = 27(1 - r) = 18$ or 9
Answer: $a = 9$ or $a = 18$

9 Equation of a cubic from its graph (All boards)

The diagram shows a sketch of the curve $y = f(x)$, where $f(x)$ is a cubic.
Find $f(x)$, giving your answer in expanded form.

Roots at $x = -4, -1, 3$, so $f(x) = k(x + 4)(x + 1)(x - 3)$
$f(0) = -12$: $k(4)(1)(-3) = -12$
$k = 1$, then expand
Answer: $f(x) = x^3 + 2x^2 - 11x - 12$

10 Foot of perpendicular and distance to a line (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ -4 \\ -5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Find the coordinates of the foot of the perpendicular from P(−1, 1, −6) to l, and the exact distance from P to l.

Foot $F = A + t\,d$: $\mathbf{PF} = \begin{pmatrix} t - 1 \\ 2t - 5 \\ 2t + 1 \end{pmatrix}$
$\mathbf{PF} \cdot \mathbf{d} = 0$ gives $9t - 9 = 0$, so $t = 1$
$F = (-1, -2, -3)$; $\mathbf{PF} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$, so $|\mathbf{PF}| = \sqrt{18} = 3\sqrt{2}$
Answer: F(−1, −2, −3), distance $3\sqrt{2}$