Day 325 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Binomial expansion: a coefficient (All boards)

Find the coefficient of x^2 in the expansion of $(1 + 3x)^7$.

Term in x^2: ${}^7C_2 \times 1^5 \times (3x)^2$

$= 21 \times 1 \times 9$

Answer: 189

2 Tree diagram with conditional probability (All boards)

The tree diagram shows probabilities for events A and B.

(a) Find P(B′).

(b) Find P(A′ | B′).

(a) $P(B') = 0.5 \times 0.3 + 0.5 \times 0.4 = 0.35$

(b) $P(A' \mid B') = \dfrac{0.5 \times 0.4}{0.35}$

Answer: (a) 0.35 (b) $\frac{4}{7}$

3 Pulled at an angle on rough ground (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 12 kg box is pulled along rough horizontal ground, $\mu = 0.1$, by a rope at angle θ above the horizontal, where $\tan\theta = \frac{5}{12}$.

The tension is 40 N. Find (a) the normal reaction, (b) the acceleration of the box. Take $g = 9.8$. Give answers to 3 s.f. if not exact.

$\sin\theta = \dfrac{5}{13}$, $\cos\theta = \dfrac{12}{13}$

(a) Vertical: $R + 40\sin\theta = 12g$, so $R = \dfrac{6644}{65}$

(b) Horizontal: $40\cos\theta - \mu R = 12a$

$\dfrac{480}{13} - 0.1 \times \dfrac{6644}{65} = 12a$

$a = \dfrac{4339}{1950}$

Answer: (a) 102 N (b) 2.23 m s^{-2}

4 Exponential growth and decay modelling (All boards)

The number of fish in a lake is modelled by $N = 100\,e^{kt}$, where t is time in years.

After 6 years, N = 600.

Find (a) the exact value of k, (b) the time for N to reach 1000, to 3 s.f.

$100\,e^{6k} = 600$, so $e^{6k} = 6$ and $k = \dfrac{\ln 6}{6}$

$k = 0.299$ (3 s.f.)

$100\,e^{kt} = 1000$ gives $kt = \ln 10$, so $t = \dfrac{\ln 10}{k} = 7.71$ years

Answer: (a) $k = \frac{\ln 6}{6}$ (b) t = 7.71 years

5 Geometric series: smallest n (All boards)

A geometric series has first term 3 and common ratio 2.

Find the smallest n for which the sum of the first n terms exceeds 10000.

$S_n = \dfrac{3(2^n - 1)}{(2 - 1)} > 10000$

$2^n > \dfrac{10003}{3}$

$n > \dfrac{\log \frac{10003}{3}}{\log 2} = 11.70$

Check: $S_{11} = 6141.0$, $S_{12} = 12285.0$

Answer: n = 12

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6 Perpendicular vectors and a constant (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The vectors $\begin{pmatrix} k \\ k \\ 5 \end{pmatrix}$ and $\begin{pmatrix} k \\ -6 \\ 1 \end{pmatrix}$ are perpendicular.

Find the possible values of k and, for the larger value, the exact length of the first vector.

Scalar product = 0: $k^2 - 6k + 5 = 0$
$(k - 5)(k - 1) = 0$

k = 1 or k = 5. For k = 5 the first vector is $\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$, with length $\sqrt{75} = 5\sqrt{3}$

Answer: k = 1 or 5; length $5\sqrt{3}$

7 Hypothesis test for a Poisson mean (CIE)

Accidents occur as a Poisson process at a mean rate of 3 per week.
After a change, the number recorded in one week is 9.
Test at the 1% level whether the mean rate has increased.

H_0: $\lambda = 3$, H_1: $\lambda > 3$; $X \sim Po(3)$
$P(X \geq 9) = 1 - P(X \leq 8) = 0.00380$
$0.00380 < 0.01$
Reject H_0: evidence the rate has increased.

Answer: p-value 0.00380; reject H_0

8 Projectile: perpendicular directions (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the ground at 10 m s^{-1} at angle α above the horizontal, $\tan\alpha = \frac{24}{7}$. At time T s, while in flight, it is moving perpendicular to its initial direction. Find (a) T, (b) the height of the ball then. Take g = 9.8. Give answers to 3 s.f. if not exact.

Initial velocity (2.8, 9.6). At time T the velocity is (2.8, 9.6 − 9.8T).
Perpendicular: scalar product = 0: $(2.8)^2 + 9.6(9.6 - 9.8T) = 0$
$10^2 - 9.8T \times 9.6 = 0$ (using $\sin^2\alpha + \cos^2\alpha = 1$)

(a) $T = 10 \div (9.8 \times 0.96) = \frac{625}{588}$

(b) $h = 9.6 \times \frac{625}{588} - 4.9 \times \left(\frac{625}{588}\right)^2 = \frac{65875}{14112}$

Answer: (a) T = 1.06 s (b) 4.67 m

9 Recurrence relations: limits (All boards)

$u_1 = 0$ and $u_{n+1} = \frac{1}{2} u_n + 2$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.01$.

(a) $L = \frac{1}{2} L + 2$ gives $L = \frac{2}{1 - \frac{1}{2}} = 4$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = (-4) \times \left(\frac{1}{2}\right)^{n-1}$

$|-4| \times \left(\frac{1}{2}\right)^{n-1} < \frac{1}{100}$

Check: n = 9 gives 0.01562, n = 10 gives 0.00781

Answer: (a) L = 4 (b) n = 10

10 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

The volume of a cube of edge x cm is increasing at 27 cm^3s^{-1}.
Find the rate of increase of its total surface area when x = 2.

$V = x^3$, $\frac{dV}{dx} = 3x^2$, so $\frac{dx}{dt} = \frac{27}{3x^2} = \frac{9}{4}$ at x = 2

$S = 6x^2$, $\frac{dS}{dx} = 12x$, so $\frac{dS}{dt} = 12 \times 2 \times \frac{9}{4} = 54$

Answer: 54 cm^2/s