Day 32 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Disproof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas)

Show by counter-example that the statement "If $x^2 > 64$ then $x > 8$" is false.

Take $x = -9$:

$x^2 = 81 > 64$, but $x = -9$ is not greater than 8.

Answer: $x = -9$

2 Perpendicular vectors and a constant (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The vectors $\begin{pmatrix} k \\ k \\ -1 \end{pmatrix}$ and $\begin{pmatrix} k \\ -3 \\ 10 \end{pmatrix}$ are perpendicular.

Find the possible values of k and, for the larger value, the exact length of the first vector.

Scalar product = 0: $k^2 - 3k - 10 = 0$

$(k + 2)(k - 5) = 0$

$k = -2$ or $k = 5$. For $k = 5$ the first vector is $\begin{pmatrix} 5 \\ 5 \\ -1 \end{pmatrix}$, with length $\sqrt{51} = \sqrt{51}$

Answer: $k = -2$ or 5; length $\sqrt{51}$

3 Partial fractions: distinct linear factors (All boards)

Express $\frac{5x + 20}{(x + 1)(2x + 5)}$ in partial fractions.

Let $\frac{5x + 20}{(x + 1)(2x + 5)} \equiv \frac{A}{(x + 1)} + \frac{B}{(2x + 5)}$

$5x + 20 \equiv A(2x + 5) + B(x + 1)$

Substitute $x = -1$: $A = 5$

Substitute $x = -\frac{5}{2}$: $B = -5$

Answer: $\frac{5}{(x + 1)} + \frac{-5}{(2x + 5)}$

4 Systematic sampling (AQA, Edexcel, OCR, MEI, Eduqas)

The 380 items in a list are numbered 1 to 380.

A systematic sample of 19 items is taken, starting with a random item among the first few.

The first item selected is number 13.

Find the sampling interval and the number of the last item selected.

Interval = 380 ÷ 19 = 20

Items selected: 13, 33, 53, ...

Last = 13 + (19 − 1) × 20 = 373

Answer: Interval 20, last item 373

5 Recurrence relations: unknown first term (All boards)

A sequence is defined by $u_1 = k$ and $u_{n+1} = 4u_n + 3$.

Given that $u_3 = 47$, find (a) the value of k, (b) $\sum_{r=1}^{4} u_r$.

$u_2 = 4k + 3$, $u_3 = 4(4k + 3) + 3$

$16k + 15 = 47$ so $k = 2$

Terms: 2, 11, 47, 191

Sum = 251

Answer: (a) $k = 2$ (b) 251

6 Parametric differentiation with trigonometry (All boards)

A curve has parametric equations $x = 2\sec t$, $y = 2\tan t$.

Find $\frac{dy}{dx}$ in terms of t and the exact equation of the tangent at the point where $t = \frac{\pi}{4}$, in the form $y = mx + c$.

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = \frac{2}{2\sin t}$

$t = \frac{\pi}{4}$: gradient $m = \sqrt{2}$, point $(2\sqrt{2}, 2)$

$c = y - mx = -2$

Answer: $\frac{dy}{dx} = \frac{2}{2\sin t}$; $y = \sqrt{2}x - 2$

7 Horizontal projection (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected horizontally at 24 m s^{-1} from a point 19.6 m above horizontal ground. Find (a) the horizontal distance to the landing point, (b) the speed and direction (angle below the horizontal, 1 d.p.) on landing. Take g = 9.8. Give answers to 3 s.f. if not exact.

Vertical: $19.6 = \frac{1}{2} \times 9.8 \times t^2$, so $t = 2$

(a) Horizontal distance = 24×2

(b) Vertical speed on landing = $9.8 \times 2 = 19.6$; horizontal speed = 24

Speed = $\sqrt{(24^2 + 19.6^2)} = 31.0$

$\tan\theta = 19.6 \div 24$, $\theta = 39.2°$

Answer: (a) 48 m (b) 31.0 m s^{-1} at 39.2° below the horizontal

8 Equations with sec x and tan x (All boards)

Solve $4\sec^2\theta - 2\tan\theta - 6 = 0$

for $0° \le \theta \le 360°$. Give answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $4\tan^2\theta - 2\tan\theta - 2 = 0$

$\tan\theta = -\frac{1}{2}$ or $\tan\theta = 1$

$\tan\theta = -\frac{1}{2}$: $\theta = 153.4°, 333.4°$

$\tan\theta = 1$: $\theta = 45.0°, 225.0°$

Answer: θ = 45.0°, 153.4°, 225.0°, 333.4°

9 Binomial mean and variance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X has a binomial distribution with mean 9 and variance 7.2.

(a) Find n and p.

(b) Find P(X = 11) to 3 significant figures.

(a) $np = 9$ and $npq = 7.2$

$q = \frac{npq}{np} = \frac{7.2}{9} = 0.8$, so $p = 0.2$

$n = \frac{9}{0.2} = 45$

(b) $P(X = 11) = C(45, 11)(0.2)^{11}(0.8)^{34} = 0.105$

Answer: (a) n = 45, p = 0.2 (b) 0.105

10 Ladder in limiting equilibrium (AQA, Edexcel, OCR, MEI)

A uniform ladder AB (15 m, 15 kg) rests with A on rough ground, $\mu = 0.2$, and B against a smooth wall, B being 12 m above the ground. A man of mass 70 kg climbs the ladder. How far along the ladder from A can he climb before it slips?

Foot of ladder is $\sqrt{(15^2 - 12^2)} = 9$ m from the wall

Vertical: $R = (15 + 70)g$. Limiting friction: $F = \mu R = 0.2(15 + 70)g$

Horizontal: $S = F$ (wall reaction S)

Moments about A (man at distance x along ladder, horizontal distance $x \times 9/15$):

$S \times 12 = 15\,g \times 4.5 + 70\,g \times (9x \div 15)$

$17 \times 12 = 67.5 + 42x$ (g cancels), so $x = 3.25$

Answer: 3.25 m