Day 31 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Smallest whole number solving an exponential inequality (All boards)

Find the least integer n for which $2^n > 10^9$.

n lg 2 > 9, so n > $\frac{9}{\lg 2}$ = 29.90

Since n must be an integer, n = 30

Answer: n = 30

2 Stratified sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A club has 280 members: 56 Bronze, 120 Silver and 104 Gold.
A stratified sample of size 35 is taken.
Find the number of members sampled from each group.

Sampling fraction = $\frac{35}{280} = \frac{1}{8}$

Bronze: $56 \times \frac{1}{8} = 7$

Silver: $120 \times \frac{1}{8} = 15$

Gold: $104 \times \frac{1}{8} = 13$

Answer: Bronze 7, Silver 15, Gold 13

3 Proof by exhaustion (AQA, Edexcel, OCR, MEI, Eduqas)

Prove that the square of any integer is either a multiple of 4 or one more than a multiple of 4.

Any integer is even (2k) or odd (2k + 1).
$(2k)^2 = 4k^2$, a multiple of 4.
$(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$
Both cases covered.
Answer: squares are of the form 4m or 4m + 1

4 Expectation and variance of linear combinations (Edexcel, OCR, MEI, Eduqas, CIE)

X has E(X) = 3 and Var(X) = 3.
(a) Find $E(X^2)$.
(b) Y = 4X + 7. Find E(Y) and Var(Y).
(c) X_1 and X_2 are independent copies of X. Find $\text{Var}(X_1 - X_2)$.

(a) $E(X^2) = \text{Var}(X) + [E(X)]^2 = 3 + 9 = 12$
(b) $E(Y) = 4 \times 3 + 7 = 19$, $\text{Var}(Y) = (4)^2 \times 3 = 48$
(c) $\text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 2 \times 3 = 6$
Answer: (a) 12 (b) E(Y) = 19, Var(Y) = 48 (c) 6

5 Passing a point twice (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is projected vertically upwards from the ground. It passes a ledge 2 s after projection and again 3 s after projection. Find (a) the speed of projection, (b) the height of the ledge, (c) the greatest height. Take g = 9.8.

Same height at t = 2 and t = 3, so the stone is at its highest point at t = 2.5
(a) At the top v = 0: $0 = u - 9.8 \times 2.5$, so u = 24.5
(b) $s = ut - 4.9t^2$ with t = 2: $s = 24.5 \times 2 - 4.9 \times 2^2 = 29.4$
(c) $v^2 = u^2 - 2gs$: $0 = 24.5^2 - 19.6s$, s = 30.625
Answer: (a) 24.5 m s^{-1} (b) 29.4 m (c) 30.625 m

6 Exponential equations reducing to quadratics (All boards)

Solve $2^{2x} - 232^x + 76 = 0$.
Give exact roots, or 3 s.f. where the answer is not exact.

Let $u = 2^x$: $u^2 - 23u + 76 = 0$, so $(u - 4)(u - 19) = 0$
$2^x = 4$ gives x = 2; $2^x = 19$ gives $x = \frac{\ln 19}{\ln 2} = 4.25$
Answer: x = 2 or x = 4.25

7 Newton's second law with a time-varying force (AQA, Edexcel, OCR, MEI)

A 2 kg particle moves on a smooth horizontal line. At time t s a single force of $(15 - 3t)$ N acts along the line. When t = 0 its speed is 1 m s^{-1}. Find (a) its speed when t = 2, (b) its greatest speed.

F = ma: $a = (-3t + 15) \div 2$

$v = \int a\,dt = 7.5t - 0.75t^2 + c$; v = 1 when t = 0

(a) v(2) = 13

(b) Greatest speed when a = 0: t = 5

v(5) = 19.75

Answer: (a) 13 m s^{-1} (b) 19.75 m s^{-1}

8 Recurrence relations: limits (All boards)

$u_1 = 0$ and $u_{n+1} = \frac{1}{4}u_n + 18$.

(a) Find the limit L of the sequence as $n \to \infty$.

(b) Find the least n for which $|u_n - L| < 0.1$.

(a) $L = \frac{1}{4}L + 18$ gives $L = \dfrac{18}{1 - \frac{1}{4}} = 24$

(b) $u_n - L = (u_1 - L)\,r^{n-1} = (-24) \times \left(\frac{1}{4}\right)^{n-1}$

$|-24| \times \left(\frac{1}{4}\right)^{n-1} < \frac{1}{10}$

Check: n = 4 gives 0.37500, n = 5 gives 0.09375

Answer: (a) L = 24 (b) n = 5

9 Implicit differentiation: products and cubes (All boards)

A curve has equation $x^2y - y^3 + x = -25$.

Find $\frac{dy}{dx}$ in terms of x and y, and the gradient of the normal at (−1, 3).

Differentiate: $2xy + x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} + 1 = 0$

$\frac{dy}{dx} = -\dfrac{2xy + 1}{x^2 - 3y^2}$

At (−1, 3): $\frac{dy}{dx} = -\frac{5}{26}$, so the normal has gradient $\frac{26}{5}$

Answer: $\frac{dy}{dx} = -\dfrac{2xy + 1}{x^2 - 3y^2}$; normal gradient $\frac{26}{5}$

10 Modulus equations (All boards)

Solve $|x^2 - 5x| = 6$.

$x^2 - 5x - 6 = 0$ gives x = −1 or x = 6

$x^2 - 5x + 6 = 0$ gives x = 2 or x = 3

Answer: x = −1, 2, 3, 6