Day 296 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Greatest and least values using R-form (All boards)

$g(\theta) = \frac{1}{16 + 5\sin\theta + 12\cos\theta}$

(a) Write $5\sin\theta + 12\cos\theta$ in the form $R\sin(\theta + \alpha)$, with α acute.

(b) Find the greatest and least values of $g(\theta)$ and the smallest positive θ at which the greatest value occurs (1 d.p.).

$R = \sqrt{5^2 + 12^2} = 13$, $\alpha = \tan^{-1}\frac{12}{5} = 67.4°$

Denominator $= 16 + 13\sin(\theta + 67.4°)$, ranging from 3 to 29

Greatest g $= \frac{1}{3}$ (denominator least), least g $= \frac{1}{29}$

Least denominator when $\sin(\theta + 67.4°) = -1$: $\theta + 67.4° = 270°$

$\theta = 202.6°$

Answer: R = 13, α = 67.4°; greatest $\frac{1}{3}$, least $\frac{1}{29}$; θ = 202.6°

2 Outliers (AQA, Edexcel, OCR, MEI, Eduqas)

A data set has mean 46 and standard deviation 9.

A value is treated as an outlier if it is more than 3 standard deviations from the mean.

Which of 22, 38, 75, 82 are outliers?

3 standard deviations $= 3 \times 9 = 27$

Lower limit $= 46 - 27 = 19$

Upper limit $= 46 + 27 = 73$

Outliers are values below 19 or above 73

Answer: 75, 82

3 Conservation of momentum (CIE)

Particles A (8 kg) and B (4 kg) move towards each other on a smooth horizontal line at 8 m s^{-1} and 7 m s^{-1}. They collide directly, and A then moves in the opposite direction at 2 m s^{-1}. Find (a) the speed of B after the collision, (b) the magnitude of the impulse of A on B.

Take the initial direction of A as positive. Conservation of momentum:

$8\times8 + 4\times(-7) = 8\times(-2) + 4v$

$36 = -16 + 4v$, so $v = 13$

(b) Impulse on B = change in momentum of B $= 4(13 - (-7))$

$= 4 \times 20$

Answer: (a) 13 m s^{-1} (b) 80 N s

4 Line and curve intersection (diagram) (All boards)

The diagram shows the curve $y = x^2 - 2x - 3$ and the line $y = -2x + 1$, meeting at A and B.

(a) Find the coordinates of A and B.

(b) Find the coordinates of the midpoint of AB.

$x^2 - 2x - 3 = -2x + 1$: $x^2 - 4 = 0$

$(x + 2)(x - 2) = 0$, $x = -2$ or 2

A(−2, 5), B(2, −3)

Midpoint: average the coordinates

Answer: (a) A(−2, 5), B(2, −3) (b) (0, 1)

5 Pulley, string goes slack (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (9 kg) and B (1 kg) hang on a light string over a smooth pulley, A being 0.5 m above the ground. Released from rest; A lands without rebounding. Find (a) the tension while both move, (b) the greatest height of B above its start. Take g = 9.8.

A: $9g - T = 9a$; B: $T - 1g = 1a$

$a = 8g \div 10 = 7.84$, so $T = 1(g + a) = 17.64$

When A lands, B has risen 0.5 m with $v^2 = 2\times7.84\times0.5 = 7.84$

B then moves freely under gravity: extra height $= v^2 \div (2\times9.8) = 0.4$

Total $= 0.5 + 0.4$

Answer: (a) 17.64 N (b) 0.9 m

6 Addition formulae with given sin and cos (All boards)

A is acute and B is obtuse, with $\sin A = \frac{7}{25}$ and $\cos B = -\frac{12}{13}$.

Find the exact values of cos(A−B) and tan(A−B).

$\cos A = \frac{24}{25}$ (A acute), $\sin B = \frac{5}{13}$ (B obtuse), so $\tan A = \frac{7}{24}$, $\tan B = -\frac{5}{12}$

$\cos(A-B) = \cos A \cos B + \sin A \sin B = \frac{24}{25} \times (-\frac{12}{13}) + \frac{7}{25} \times \frac{5}{13} = -\frac{253}{325}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{7}{24} - (-\frac{5}{12})}{1 + \frac{7}{24} \times (-\frac{5}{12})} = \frac{204}{253}$

Answer: $\cos(A-B) = -\frac{253}{325}$, $\tan(A-B) = \frac{204}{253}$

7 Estimating mean and standard deviation (All boards)

The table summarises the height, h cm, of seedlings.

Estimate the mean and standard deviation (3 s.f.).

Midpoints: 22.5, 27.5, 32.5, 37.5, 42.5 (n = 54)

$\Sigma fx = 1610$, so mean $= \frac{1610}{54} = 29.8$

$\Sigma fx^2 = 49587.5$

$\text{sd} = \sqrt{\frac{49587.5}{54} - (29.8)^2} = 5.42$

Answer: mean 29.8, sd 5.42

8 Circle and line: chord length (AQA, Edexcel, OCR, MEI, Eduqas)

The line $-3x + 4y = -14$ meets the circle $(x-1)^2 + (y+2)^2 = 25$ at the points A and B. Find the exact length of AB.

Centre (1, −2), radius 5. Distance from centre to line:

$d = \frac{|3|}{5} = \frac{3}{5}$

$\text{Half-chord}^2 = 25 - (\frac{3}{5})^2 = \frac{616}{25}$

$AB = 2\sqrt{\frac{616}{25}}$

Answer: $\frac{20\sqrt{154}}{25}$

9 Angle between vectors (3D) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The points A, B and C have coordinates (0, 2, 4), (0, 2, 2) and (4, 0, −2).

Find angle ABC, in degrees to 1 d.p.

$\text{BA} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\text{BC} = \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}$

$\text{BA} \cdot \text{BC} = -8$

$|\text{BA}| = \sqrt{4}$, $|\text{BC}| = \sqrt{36}$

$\cos \text{ABC} = \frac{-8}{\sqrt{4} \times \sqrt{36}} = -0.6667$, so angle ABC = 131.8°

Answer: 131.8°

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10 Partial fractions and binomial expansion (All boards)

$f(x) = \frac{12x + 6}{(1 + 3x)(1 + x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) up to x^2.

(c) State the range of x for which the expansion is valid.

(a) $f(x) = \frac{A}{(1 + 3x)} + \frac{B}{(1 + x)}$: A = 3, B = 3

(b) $3(1 + 3x)^{-1} = 3(1 - 3x + 9x^2 ...)$

Add the series for both fractions.

(c) Need $|3x| < 1$ and $|x| < 1$

Answer: (a) $\frac{3}{(1 + 3x)} + \frac{3}{(1 + x)}$ **(b)** $6 - 12x + 30x^2$ **(c)** $|x| < \frac{1}{3}$