Day 293 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^2 - 4$ and $g(x) = 2x - 4$.
Solve $fg(x) = gf(x)$.

$fg(x) = (2x - 4)^2 - 4 = 4x^2 - 16x + 12$
$gf(x) = 2(x^2 - 4) - 4 = 2x^2 - 12$
$4x^2 - 16x + 12 = 2x^2 - 12$ so $2x^2 - 16x + 24 = 0$
Answer: $x = 2$ or $x = 6$

2 Index equations solved with logarithms (All boards)

Solve $7^{2x-3} = 3^{3x+1}$.
Give your answer to 3 significant figures.

Take logs: $(2x - 3)\ln 7 = (3x + 1)\ln 3$
Collect x terms: $x(2\ln 7 - 3\ln 3) = \ln 3 + 3\ln 7$
$x = \dfrac{\ln 3 + 3\ln 7}{2\ln 7 - 3\ln 3}$
$x = 11.6$
Answer: $x = 11.6$

3 Symmetric interval about the mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the mass (g), is $N(59, 5^2)$.
Find the symmetric limits about the mean between which the central 99% of values lie (3 s.f.).

Each tail has area 0.005, so $P(Z < z) = 0.995$
$z = 2.58$
Limits: $59 \pm 2.58 \times 5$
$= 46.1$ and 71.9
Answer: 46.1 to 71.9

4 Newton-Raphson method (AQA, Edexcel, OCR, MEI, Eduqas)

$f(x) = x^2 - 3$. Show that the Newton-Raphson formula gives $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$,

and use it with $x_0 = 2$ to find x_2 as an approximation to $\sqrt{3}$, to 4 d.p.

$f'(x) = 2x$, so $x_{n+1} = x_n - \dfrac{x_n^2 - 3}{2x_n} = \frac{1}{2}(x_n + \frac{3}{x_n})$
$x_1 = \frac{1}{2}(2 + \frac{3}{2}) = \frac{7}{4}$
$x_2 = 1.7321$
Answer: $x_2 = 1.7321$

5 Factor theorem: roots with surds (All boards)

$f(x) = x^3 - 5x^2 + 4$.
(a) Show that $(x - 1)$ is a factor of $f(x)$.
(b) Solve $f(x) = 0$, giving surds in simplest form.

$f(1) = 1 - 5 + 4 = 0$, so $(x - 1)$ is a factor.
$f(x) = (x - 1)(x^2 - 4x - 4)$
$x^2 - 4x - 4 = 0$ gives $x = \dfrac{4 \pm \sqrt{32}}{2}$
$\sqrt{32} = 4\sqrt{2}$
Answer: $x = 1$ or $x = 2 \pm 2\sqrt{2}$

6 Hypothesis test for a normal mean (All boards)

The mean fill volume of bottles (ml) is normally distributed with standard deviation 10; it was previously 20. A random sample of 25 has mean 15.6. Test, at the 5% level, whether the mean has changed.

H_0: $\mu = 20$, H_1: $\mu \neq 20$

$\bar{X} \sim N(20, \frac{10^2}{25})$, $z = \frac{15.6 - 20}{\frac{10}{5}} = -2.20$

p-value = 0.0278 (two-tailed) < 0.05

Reject H_0: evidence that the mean has changed.

Answer: $z = -2.20$, $p = 0.0278$; reject H_0

7 Beam about to tilt (AQA, Edexcel, OCR, MEI)

A uniform plank AB (mass 12 kg, length 4 m) rests horizontally on supports at C and D, where AC = 1.5 m and DB = 1 m. A 5 kg particle is placed at E, AE = x m, and the plank is about to tilt about C. Find x.

About to tilt about C, so the reaction at D is zero and E lies between A and C.

Moments about C: $5\,g \times (1.5 - x) = 12\,g \times (2 - 1.5)$

$5(1.5 - x) = 6$

$x = 1.5 - 1.2$

Answer: $x = 0.3$

8 Circle and line: chord length (AQA, Edexcel, OCR, MEI, Eduqas)

The line $4x - 3y = 21$ meets the circle $(x - 1)^2 + (y + 3)^2 = 16$ at the points A and B. Find the exact length of AB.

Centre $(1, -3)$, radius 4. Distance from centre to line:

$d = \frac{|-8|}{5} = \frac{8}{5}$

$\text{Half-chord}^2 = 16 - (\frac{8}{5})^2 = \frac{336}{25}$

$AB = 2\sqrt{\frac{336}{25}}$

Answer: $\frac{40\sqrt{21}}{25}$

9 Range of horizontal force on a rough slope (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 5 kg particle is in equilibrium on a rough plane inclined at α, $\tan\alpha = \frac{3}{4}$, $\mu = 0.2$. A horizontal force of P N, in the vertical plane through a line of greatest slope, pushes it towards the plane. Find the least and greatest values of P. Take g = 9.8. Give answers to 3 s.f. if not exact.

$\sin\alpha = 0.6$, $\cos\alpha = 0.8$, weight $W = 49$ N. Normal reaction $R = W\cos\alpha + P\sin\alpha$

Least P: about to slide down, friction $F = \mu R$ up the slope

$P\cos\alpha + \mu(W\cos\alpha + P\sin\alpha) = W\sin\alpha \rightarrow P = W(\sin\alpha - \mu\cos\alpha) \div (\cos\alpha + \mu\sin\alpha) = 23.4$

Greatest P: about to slide up, friction down the slope

$P\cos\alpha = W\sin\alpha + \mu(W\cos\alpha + P\sin\alpha) \rightarrow P = W(\sin\alpha + \mu\cos\alpha) \div (\cos\alpha - \mu\sin\alpha) = 54.8$

Answer: least 23.4 N, greatest 54.8 N

10 Double and triple angle values (All boards)

Given that $\cos\theta = \frac{3}{5}$, find the exact value of $\cos 3\theta$.

$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$= 4 \times \frac{3}{5}^3 - 3 \times \frac{3}{5} = -\frac{117}{125}$

Answer: $-\frac{117}{125}$