Day 290 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Exponential growth and decay modelling (All boards)

The number of fish in a lake is modelled by $N = 400\,e^{kt}$, where t is time in years.
After 4 years, N = 2400.
Find (a) the exact value of k, (b) the time for N to reach 20000, to 3 s.f.

$400\,e^{4k} = 2400$, so $e^{4k} = 6$ and $k = \frac{\ln 6}{4}$

$k = 0.448$ (3 s.f.)

$400\,e^{kt} = 20000$ gives $kt = \ln 50$, so $t = \frac{\ln 50}{k} = 8.73$ years

Answer: (a) $k = \frac{\ln 6}{4}$ (b) t = 8.73 years

2 Arithmetic series: finding n (All boards)

An arithmetic series has first term −6 and common difference 3.
The sum of the first n terms is 126. Find n.

$\frac{n}{2}(2 \times (-6) + (n - 1) \times 3) = 126$

$3n^2 - 15n = 252$

$3n^2 - 15n - 252 = 0$

$(n - 12)(3n + 21) = 0$, so $n = 12$ or $n = -7$

n must be a positive integer, so n = 12

Answer: 12

3 Passing a point twice (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is projected vertically upwards from the ground. It passes a ledge 1 s after projection and again 8 s after projection. Find (a) the speed of projection, (b) the height of the ledge, (c) the greatest height. Take g = 9.8.

Same height at t = 1 and t = 8, so the stone is at its highest point at t = 4.5

(a) At the top v = 0: $0 = u - 9.8 \times 4.5$, so $u = 44.1$

(b) $s = ut - 4.9t^2$ with t = 1: $s = 44.1 \times 1 - 4.9 \times 1^2 = 39.2$

(c) $v^2 = u^2 - 2gs$: $0 = 44.1^2 - 19.6s$, $s = 99.225$

Answer: (a) 44.1 m s^{-1} (b) 39.2 m (c) 99.225 m

4 Change of sign to locate a root (All boards)

$f(x) = x^3 - 2x - 11$.
Show that the equation f(x) = 0 has a root α in the interval [2, 3].

f(x) is continuous on the interval

$f(2) = -7.00 < 0$

$f(3) = 10.0 > 0$

A change of sign, so a root lies in [2, 3]

Answer: Sign change: f(2) = −7.00, f(3) = 10.0

5 Projected up a rough incline (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 12.6 m s^{-1} up a line of greatest slope of a rough plane inclined at α, where $\tan\alpha = \frac{12}{5}$. The coefficient of friction is 0.3. Find (a) the greatest distance it moves up the plane, (b) its speed on returning to the start. Take g = 9.8. Give answers to 3 s.f. if not exact.

Up: deceleration $= g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{12}{13} + 0.3 \times \frac{5}{13}) = \frac{1323}{130}$

(a) $0 = 12.6^2 - 2 \times \frac{1323}{130} \times d$, so $d = 7.8$

Down: $\tan\alpha = \frac{12}{5} > \mu$, so it slides back; acceleration $= g(\sin\alpha - \mu\cos\alpha) = \frac{1029}{130}$

(b) $v^2 = 2 \times \frac{1029}{130} \times 7.8 = 123.48$

$v = \sqrt{123.48}$

Answer: (a) 7.8 m (b) 11.1 m s^{-1}

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6 Partial fractions: repeated factor (All boards)

Express $\frac{-6x^2 + 22x - 5}{(x + 3)(x - 2)^2}$ in the form $\frac{A}{(x + 3)} + \frac{B}{(x - 2)} + \frac{C}{(x - 2)^2}$, and find A, B and C.

$-6x^2 + 22x - 5 \equiv A(x - 2)^2 + B(x + 3)(x - 2) + C(x + 3)$

$x = -3$: $-125 = A(-5)^2$ so $A = -5$

$x = 2$: $15 = C(5)$ so $C = 3$

Coefficient of x^2: $-6 = A + B$ so $B = -1$

Answer: $A = -5$, $B = -1$, $C = 3$

7 Combining two data sets (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The times in seconds of two groups: group X has n = 9, mean 68, sd 9;
group Y has n = 20, mean 39, sd 9.
Find the mean and standard deviation of all 29 values (3 s.f. for the sd).

$\Sigma x = 9 \times 68 + 20 \times 39 = 1392$, mean = 48

Σx^2 (A) $= 9(9^2 + 68^2) = 42345$

Σx^2 (B) $= 20(9^2 + 39^2) = 32040$

Variance $= \frac{74385}{29} - 48^2 = 261$, so sd = 16.2

Answer: mean 48, sd 16.2

8 Equations with sin 2x and a trig function (All boards)

Solve $\sin 2\theta = \cos\theta$ for $0 \le \theta \le 2\pi$.

$\sin 2\theta \equiv 2\sin\theta\cos\theta$: $\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin\theta = \frac{1}{2}$: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Do not divide by $\cos\theta$, as that would lose solutions

Answer: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

9 Type I and Type II errors (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(33, p)$. H_0: $p = 0.4$ is tested against H_1: $p < 0.4$ using a 5% significance level.
(a) Find the critical region. (b) State the probability of a Type I error.
(c) Find the probability of a Type II error when $p = 0.2$ (3 s.f.).

(a) Largest/smallest region with probability ≤ 0.05 under H_0: $X \le 8$

(b) Type I error = P(reject H_0 | $p = 0.4$) = 0.0444

(c) Type II error = P(not in critical region | $p = 0.2$)

$P(X \ge 9)$ when $p = 0.2$: 0.200

Answer: (a) $X \le 8$ (b) 0.0444 (c) 0.200

10 Partial fractions and binomial expansion (All boards)

$f(x) = \frac{6x + 7}{(1 - 2x)(1 + 3x)}$.
(a) Express f(x) in partial fractions.
(b) Hence expand f(x) up to x^2.
(c) State the range of x for which the expansion is valid.

(a) $f(x) = \frac{A}{(1 - 2x)} + \frac{B}{(1 + 3x)}$: $A = 4$, $B = 3$

(b) $4(1 - 2x)^{-1} = 4(1 + 2x + 4x^2 \ldots)$

Add the series for both fractions.

(c) Need $|2x| < 1$ and $|3x| < 1$

Answer: (a) $\frac{4}{(1 - 2x)} + \frac{3}{(1 + 3x)}$ (b) $7 - x + 43x^2$ (c) $|x| < \frac{1}{3}$