Day 29 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Systematic sampling (AQA, Edexcel, OCR, MEI, Eduqas)

The 600 items in a list are numbered 1 to 600.
A systematic sample of 30 items is taken, starting with a random item among the first few.
The first item selected is number 15.
Find the sampling interval and the number of the last item selected.

Interval = 600 ÷ 30 = 20
Items selected: 15, 35, 55, ...
Last = 15 + (30 − 1) × 20 = 595
Answer: Interval 20, last item 595

2 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x + 4$ and $g(x) = x^2 - 2$.
Solve $gf(x) = 2$.

$gf(x) = g(2x + 4) = (2x + 4)^2 - 2$
$(2x + 4)^2 - 2 = 2$
$2x + 4 = \pm 2$
Answer: $x = -3$ or $x = -1$

3 Exact definite integrals of standard functions (All boards)

Find the exact value of

$$\int_0^{\ln 3} (4e^{2x} + 4e^x)\,dx.$$

Integrate: $[2e^{2x} + 4e^x]$
Substitute the limits and subtract: 24
Answer: 24

4 Sigma notation (All boards)

Find the value of $\sum_{r=1}^{8} 5 \times 4^r$.

Geometric series: first term 20, ratio 4, 8 terms

$$S = \frac{20(4^8 - 1)}{(4 - 1)}$$

$$= \frac{20 \times 65535}{3}$$

Answer: 436900

5 Greatest and least values using R-form (All boards)

$$g(x) = \frac{1}{17 + 5\sin x + 12\cos x}$$

(a) Write $5\sin x + 12\cos x$ in the form $R\sin(x + \alpha)$, with α acute.
(b) Find the greatest and least values of g(x) and the smallest positive x at which the greatest value occurs (1 d.p.).

$R = \sqrt{5^2 + 12^2} = 13$, $\alpha = \tan^{-1}\frac{12}{5} = 67.4°$

Denominator = $17 + 13\sin(x + 67.4°)$, ranging from 4 to 30

Greatest $g = \frac{1}{4}$ (denominator least), least $g = \frac{1}{30}$

Least denominator when $\sin(x + 67.4°) = -1$: $x + 67.4° = 270°$
$x = 202.6°$

Answer: $R = 13$, $\alpha = 67.4°$; greatest $\frac{1}{4}$, least $\frac{1}{30}$; $x = 202.6°$

6 Asymptotes after transformation (All boards)

The curve y = f(x), where f(x) = $\frac{1}{x+1}$ + 2, is translated by $\begin{pmatrix}-3\\1\end{pmatrix}$.

Write down the equations of the asymptotes of the new curve, and find where it crosses the x-axis.

Asymptotes of y = f(x): x = −1, y = 2

Translate by $\begin{pmatrix}-3\\1\end{pmatrix}$: x = −4, y = 3

New curve: y = $\frac{1}{x+4}$ + 3

y = 0: $\frac{1}{x+4}$ = −3, so x + 4 = $-\frac{1}{3}$

Answer: x = −4, y = 3; crosses x-axis at $(-\frac{13}{3}, 0)$

7 Projected up a rough incline (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 6.3 m s^{-1} up a line of greatest slope of a rough plane inclined at α, where tan α = $\frac{12}{5}$. The coefficient of friction is 0.3. Find (a) the greatest distance it moves up the plane, (b) its speed on returning to the start. Take g = 9.8. Give answers to 3 s.f. if not exact.

Up: deceleration = g(sin α + μ cos α) = 9.8($\frac{12}{13}$ + 0.3×$\frac{5}{13}$) = $\frac{1323}{130}$

(a) $0 = 6.3^2 - 2\times\frac{1323}{130}\times d$, so d = 1.95

Down: tan α = $\frac{12}{5}$ > μ, so it slides back; acceleration = g(sin α − μ cos α) = $\frac{1029}{130}$

(b) $v^2 = 2\times\frac{1029}{130}\times 1.95 = 30.87$

v = √30.87

Answer: (a) 1.95 m (b) 5.56 m s^{-1}

8 Conditional binomial probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(11, 0.5). Find P(X = 5 | X ≥ 3) to 3 significant figures.

P(X = 5 | X ≥ 3) = $\frac{P(X=5)}{P(X\geq 3)}$ (since X = 5 implies X ≥ 3)

P(X = 5) = 0.226

P(X ≥ 3) = 1 − P(X ≤ 2) = 0.967

Ratio = 0.233

Answer: 0.233

9 Intersecting or skew lines (CIE (9709); Further Maths on other boards)

l_1: $\mathbf{r} = \begin{pmatrix}4\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\2\end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix}-3\\-3\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\0\end{pmatrix}$

Determine whether l_1 and l_2 intersect. If they do, find the point of intersection.

Equate components: 4 + 3λ = −3 + 1μ; 1 + 2λ = −3 + 0μ; 3 + 2λ = −1 + 0μ

Solving: λ = −2, μ = 1, and all three equations are satisfied

Point of intersection = (−2, −3, −1)

Answer: Intersect at (−2, −3, −1)

10 Ladder in limiting equilibrium (AQA, Edexcel, OCR, MEI)

A uniform ladder AB (5 m, 30 kg) rests with A on rough ground, $\mu = 0.4$, and B against a smooth wall, B being 3 m above the ground. A man of mass 60 kg climbs the ladder. How far along the ladder from A can he climb before it slips?

Foot of ladder is $\sqrt{(5^2 - 3^2)} = 4$ m from the wall
Vertical: $R = (30 + 60)g$. Limiting friction: $F = \mu R = 0.4(30 + 60)g$
Horizontal: $S = F$ (wall reaction S)
Moments about A (man at distance x along ladder, horizontal distance $x \times 4/5$):
$S \times 3 = 30\,g \times 2 + 60\,g \times (4x \div 5)$
$36 \times 3 = 60 + 48x$ (g cancels), so $x = 1$
Answer: 1 m