Day 289 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Cubic inequalities (All boards)

Given that $(x - 5)$ is a factor of $f(x) = x^3 - 21x - 20$, solve $f(x) > 0$.

Factorise: $f(x) = (x + 4)(x + 1)(x - 5)$
Roots at $x = -4, -1, 5$
Sketch the cubic (positive x^3 coefficient: negative, positive, negative, positive)
Read off where $f(x) > 0$.
Answer: $-4 < x < -1$ or $x > 5$

2 Smooth incline: passing a point twice (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected up a line of greatest slope of a smooth plane inclined at α. It passes A, 4.9 m from the start, after 1 s and again after 4 s. Find (a) the speed of projection, (b) α to 1 d.p. Take $g = 9.8$.

Deceleration $a = g \sin \alpha$ (unknown). $s = ut - \frac{1}{2}at^2$ at both times:
$u \times 1 - \frac{1}{2}a \times 1^2 = u \times 4 - \frac{1}{2}a \times 4^2 = 4.9$
Subtracting: $u(4 - 1) = \frac{1}{2}a(16 - 1)$, so $u = \frac{1}{2}a(5)$
Substituting: $\frac{1}{2}a(5)(1) - \frac{1}{2}a \times 1^2 = 4.9$, so $\frac{1}{2}a \times 4 = 4.9$, $a = 2.45$
(a) $u = 1.225 \times 5 = 6.125$
(b) $9.8 \sin \alpha = 2.45$, $\sin \alpha = 0.25$, $\alpha = 14.5°$
Answer: (a) 6.125 m s^{-1} (b) 14.5°

3 Rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas)

Express $\frac{1 + \sqrt{6}}{2 + \sqrt{6}}$ in the form $p + q\sqrt{6}$, where p and q are rational numbers.

Multiply top and bottom by $2 - \sqrt{6}$.
Denominator: $(2 + \sqrt{6})(2 - \sqrt{6}) = -2$
Numerator: $(1 + \sqrt{6})(2 - \sqrt{6}) = -4 + \sqrt{6}$
So the expression is $\frac{-4 + \sqrt{6}}{-2}$.
Answer: $2 - \frac{1}{2}\sqrt{6}$

4 Differentiating a^x (All boards)

$y = 2 \times 5^x$.
The gradient of the curve at P is $50 \ln 5$. Find the x-coordinate of P.

$\frac{dy}{dx} = 2 \ln 5 \times 5^x$
$2 \ln 5 \times 5^x = 50 \ln 5$, so $5^x = 25$
$x = 2$, so $x = 2$
Answer: $x = 2$

5 Stopping distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A driver at constant speed sees a hazard 63.75 m ahead. After a reaction time of 0.5 s she brakes with deceleration 2 m s^{-2} and stops exactly at the hazard. Find (a) her initial speed, (b) the total time from seeing the hazard to stopping.

Let the initial speed be v. Reaction distance $= 0.5v$, braking distance $= v^2 \div (2 \times 2)$
$0.5v + v^2 \div 4 = 63.75$, so $v^2 + 2v - 255 = 0$
$(v - 15)(v + 17) = 0$, so $v = 15$
(b) Braking time $= 15 \div 2 = 7.5$ s; total $= 0.5 + 7.5$
Answer: (a) 15 m s^{-1} (b) 8 s

6 R-form and solving a sin x + b cos x = c (All boards)

(a) Express 12 sin θ + 5 cos θ in the form R sin(θ + α), where R > 0 and 0° < α < 90°. Give α to 1 d.p.
(b) Solve 12 sin θ + 5 cos θ = 3 for 0° ≤ θ < 360°, giving answers to 1 d.p.

$R^2 = 12^2 + 5^2 = 169$, so R = 13; tan α = $\frac{5}{12}$ gives α = 22.6°

13 sin(θ + 22.6°) = 3, so sin(θ + 22.6°) = $\frac{3}{13}$

Principal value 13.3°, then use the symmetry of the sine graph
θ = 144.0°, 350.7°

Answer: (a) R = 13, α = 22.6° (b) θ = 144.0°, 350.7°

7 Type I and Type II errors (AQA, Edexcel, OCR, MEI, Eduqas)

X ~ B(36, p). H_0: p = 0.3 is tested against H_1: p < 0.3 using a 5% significance level.
(a) Find the critical region. (b) State the probability of a Type I error.
(c) Find the probability of a Type II error when p = 0.15 (3 s.f.).

(a) Largest/smallest region with probability ≤ 0.05 under H_0: X ≤ 5
(b) Type I error = P(reject H_0 | p = 0.3) = 0.0216
(c) Type II error = P(not in critical region | p = 0.15)
P(X ≥ 6) when p = 0.15: 0.459

Answer: (a) X ≤ 5 (b) 0.0216 (c) 0.459

8 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas)

A circle touches both coordinate axes and passes through the point (1, −8). Its centre lies in the fourth quadrant. Find the equations of the two possible circles.

Centre (r, −r), radius r > 0.
Through (1, −8): $(1 - r)^2 + (-8 + r)^2 = r^2$
$r^2 - 18r + 65 = 0$
(r − 5)(r − 13) = 0, so r = 5 or 13

Answer: $(x - 5)^2 + (y + 5)^2 = 25$ or $(x - 13)^2 + (y + 13)^2 = 169$

9 Continuous random variable: cumulative distribution and median (CIE)

The random variable X has pdf f(x) = $\frac{2}{25}(5 - x)$ for 0 ≤ x ≤ 5, and 0 otherwise.
(a) Find the cumulative distribution function F(x) for 0 ≤ x ≤ 5.
(b) Find P(X > $\frac{5}{2}$). (c) Find the median of X to 3 s.f.

(a) $F(x) = \int_0^x f(t)\,dt = \frac{2x}{5} - \frac{x^2}{25}$
(b) $P(X > \frac{5}{2}) = 1 - F(\frac{5}{2}) = \frac{1}{4}$
(c) $F(m) = \frac{1}{2}$: $\frac{2m}{5} - \frac{m^2}{25} = \frac{1}{2}$, so $m = 5(1 - \frac{1}{\sqrt{2}})$
m = 1.46

Answer: (b) $\frac{1}{4}$ (c) 1.46

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10 Implicit differentiation: products and cubes (All boards)

A curve has equation $3x^2y + 2y^3 + 2x = -7$.

Find $\frac{dy}{dx}$ in terms of x and y, and the gradient of the normal at $(-1, -1)$.

Differentiate: $6xy + 3x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx} + 2 = 0$

$\frac{dy}{dx} = -\frac{6xy + 2}{3x^2 + 6y^2}$

At $(-1, -1)$: $\frac{dy}{dx} = -\frac{8}{9}$, so the normal has gradient $\frac{9}{8}$

Answer: $\frac{dy}{dx} = -\frac{6xy + 2}{3x^2 + 6y^2}$; normal gradient $\frac{9}{8}$