Day 281 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Projected up a rough incline (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 4.2 m s^{-1} up a line of greatest slope of a rough plane inclined at α, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction is 0.5. Find (a) the greatest distance it moves up the plane, (b) its speed on returning to the start. Take g = 9.8. Give answers to 3 s.f. if not exact.

Up: deceleration = $g(\sin\alpha + \mu\cos\alpha) = 9.8(0.6 + 0.5\times0.8) = 9.8$
(a) $0 = 4.2^2 - 2\times9.8\times d$, so $d = 0.9$

Down: $\tan\alpha = \frac{3}{4} > \mu$, so it slides back; acceleration = $g(\sin\alpha - \mu\cos\alpha) = 1.96$

(b) $v^2 = 2\times1.96\times0.9 = 3.528$
$v = \sqrt{3.528}$
Answer: (a) 0.9 m (b) 1.88 m s^{-1}

2 Normal distribution probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the score (marks), is normally distributed with mean 58 and standard deviation 12.
Find (a) P(X > 85.6) (b) P(85.6 < X < 103.6), each to 3 s.f.

(a) $z = \frac{85.6 - 58}{12} = 2.30$, $P(Z > 2.30) = 0.0107$

(b) $z_1 = 2.30$, $z_2 = \frac{103.6 - 58}{12} = 3.80$
$P(85.6 < X < 103.6) = \Phi(3.80) - \Phi(2.30) = 0.0107$
Answer: (a) 0.0107 (b) 0.0107

3 Projectile from and to the ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 37 m s^{-1} at angle α above the horizontal, $\tan\alpha = \frac{12}{5}$, and lands at A. Find (a) the time of flight, (b) the greatest height, (c) OA. Take g = 9.8. Give answers to 3 s.f. if not exact.

Horizontal component = $37\times\frac{5}{13} = \frac{185}{13}$; vertical component = $37\times\frac{12}{13} = \frac{444}{13}$

(a) Vertical: $0 = \frac{444}{13}t - 4.9t^2$, so $t = \frac{4440}{637}$

(b) At the top: $0 = \left(\frac{444}{13}\right)^2 - 2\times9.8\times H$, so $H = \frac{492840}{8281}$

(c) OA = $\frac{185}{13} \times \frac{4440}{637}$
Answer: (a) 6.97 s (b) 59.5 m (c) 99.2 m

4 Modulus inequalities (All boards)

Solve $|x - 7| > |3x + 1|$.

Square both sides: $(x - 7)^2 > (3x + 1)^2$
$((x - 7) - (3x + 1))((x - 7) + (3x + 1)) > 0$
$(-2x - 8)(4x - 6) > 0$

Critical values $x = -4$ and $x = \frac{3}{2}$; sketch the (positive or negative) quadratic.

Answer: $-4 < x < \frac{3}{2}$

5 Equations with sec x and tan x (All boards)

Solve $3\sec^2\theta - 15\tan\theta + 9 = 0$
for $0° \le \theta \le 180°$. Give answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $3\tan^2\theta - 15\tan\theta + 12 = 0$
$\tan\theta = 1$ or $\tan\theta = 4$
$\tan\theta = 1$: $\theta = 45.0°$
$\tan\theta = 4$: $\theta = 76.0°$
Answer: θ = 45.0°, 76.0°

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6 Binomial expansion: rational powers (All boards)

Expand $(1 + 4x)^{-2}$ in ascending powers of x up to and including the term in x^3.
State the range of values of x for which the expansion is valid.

$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \ldots$ with $n = -2$, $y = 4x$

Coefficients of 1, x, x^2, x^3: 1, −8, 48, −256

Valid when $|4x| < 1$, i.e. $|x| < \frac{1}{4}$

Answer: $1 - 8x + 48x^2 - 256x^3$; $|x| < \frac{1}{4}$

7 Stationary points with trigonometric and exponential terms (All boards)

$y = e^{2x} - 10e^x + 12x$.

Show that the stationary points occur where $e^x = 2$ and $e^x = 3$, and find the exact y-coordinate of each, with its nature.

$\frac{dy}{dx} = 2e^{2x} - 10e^x + 12 = 2(e^x - 2)(e^x - 3)$

So $e^x = 2$ or $e^x = 3$, i.e. $x = \ln 2$ or $x = \ln 3$

$\frac{d^2y}{dx^2} = 4e^{2x} - 10e^x = e^x(4e^x - 10)$ changes sign: negative at the smaller root, positive at the larger

Answer: $x = \ln 2$, $y = 12 \ln 2 - 16$ (maximum); $x = \ln 3$, $y = 12 \ln 3 - 21$ (minimum)

8 Poisson over different intervals (CIE)

Buses arrive at a stop at random at an average of 6 per hour.
Find the probability that in a 30-minute period there are (a) zero events (b) at least 3.
(c) Find the probability that at least one occurs in each of two separate 30-minute periods.

$\lambda = 6 \times \frac{30}{60} = 3$, $X \sim Po(3)$

(a) $P(X = 0) = e^{-3} = 0.0498$
(b) $P(X \geq 3) = 1 - P(X \leq 2) = 0.577$
(c) $(1 - 0.0498)^2 = 0.903$
Answer: (a) 0.0498 (b) 0.577 (c) 0.903

9 Recurrence relations: limits (All boards)

$u_1 = 0$ and $u_{n+1} = -\frac{1}{2}u_n + 30$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.01$.

(a) $L = -\frac{1}{2}L + 30$ gives $L = \frac{30}{1 - (-\frac{1}{2})} = 20$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = (-20) \times (-\frac{1}{2})^{n-1}$

$|-20| \times \frac{1}{2}^{n-1} < \frac{1}{100}$

Check: n = 11 gives 0.01953, n = 12 gives 0.00977
Answer: (a) L = 20 (b) n = 12

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10 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially h = 36, and after 6 hours h = 25.

Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\frac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\, dh = \int -k\, dt$, so $2\sqrt{h} = -kt + C$

t = 0: $C = 2 \times 6 = 12$, so $2\sqrt{h} = 12 - kt$

t = 6, h = 25: $10 = 12 - 6k$, so $k = \frac{1}{3}$

Empty when h = 0: $kt = 12$, so $t = \frac{12}{\frac{1}{3}} = 36$ hours

Answer: 36 hours