Day 267 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Disproof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas)

Show by counter-example that the statement "$n^2 + n + 17$ is prime for all positive integers n" is false.

Try n = 16:

$16^2 + 16 + 17 = 289 = 17^2$

This is not prime.

Answer: n = 16 gives $289 = 17^2$, not prime

2 Vertical motion under gravity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards at 39.2 m s^{-1} from a point h m above horizontal ground, and hits the ground 12 s later. Find (a) h, (b) the greatest height above the ground. Take g = 9.8.

(a) Take upwards as positive: s = −h, u = 39.2, a = −9.8, t = 12

$-h = 39.2\times12 - 4.9\times12^2$, so h = 235.2

(b) At the highest point v = 0: $0 = 39.2^2 - 2\times9.8\times s$

s = 78.4 m above the point of projection

Greatest height above the ground = 235.2 + 78.4

Answer: (a) 235.2 m (b) 313.6 m

3 Newton-Raphson method (AQA, Edexcel, OCR, MEI, Eduqas)

$f(x) = x^2 - 6$. Show that the Newton-Raphson formula gives $x_{n+1} = \frac{1}{2}(x_n + \frac{6}{x_n})$,

and use it with $x_0 = 3$ to find x_2 as an approximation to $\sqrt{6}$, to 4 d.p.

$f'(x) = 2x$, so $x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{1}{2}(x_n + \frac{6}{x_n})$

$x_1 = \frac{1}{2}(3 + \frac{6}{3}) = \frac{5}{2}$

$x_2 = 2.4500$

Answer: $x_2 = 2.4500$

4 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has $P(X = x) = k(6 - x)$ for x = 0, 1, 2, 3, 4.

(a) Show that $k = \frac{1}{20}$. (b) Find E(X) and Var(X).

(c) Find $P(X \geq 2)$.

(a) $\Sigma P(X = x) = 1$: 6 + 5 + 4 + 3 + 2 = 20, so 20k = 1

(b) $E(X) = 1\times\frac{1}{4} + 2\times\frac{1}{5} + 3\times\frac{3}{20} + 4\times\frac{1}{10} = \frac{3}{2}$

$E(X^2) = 4$, so $Var(X) = 4 - (\frac{3}{2})^2 = \frac{7}{4}$

(c) $P(X \geq 2) = \frac{1}{5} + \frac{3}{20} + \frac{1}{10} = \frac{9}{20}$

Answer: (b) $E(X) = \frac{3}{2}$, $Var(X) = \frac{7}{4}$ (c) $\frac{9}{20}$

5 Angle between vectors (3D) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The points A, B and C have coordinates (3, 5, 1), (−1, −3, −3) and (−2, −3, 1).

Find angle ABC, in degrees to 1 d.p.

$BA = \begin{pmatrix}4\\8\\4\end{pmatrix}$, $BC = \begin{pmatrix}-1\\0\\4\end{pmatrix}$

$BA\cdot BC = 12$

$|BA| = \sqrt{96}$, $|BC| = \sqrt{17}$

$\cos ABC = \frac{12}{\sqrt{96}\times\sqrt{17}} = 0.2970$, so angle ABC = 72.7°

Answer: 72.7°

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6 Hypothesis test for correlation (AQA, Edexcel, OCR, MEI, Eduqas)

For a sample of 11 pairs of values of time spent on social media and exam score, r = 0.605.
The critical value for a two-tailed test at the 1% level is 0.7348.
Test, at this level, whether there is any correlation between the two variables.

H_0: $\rho = 0$, H_1: $\rho \neq 0$

Critical value 0.7348

$|r| = 0.605 < 0.7348$

This is not in the critical region, so do not reject H_0.

There is insufficient evidence of correlation.

Answer: Do not reject H_0: insufficient evidence

7 Connected rates of change: cones and bowls (AQA, Edexcel, OCR, MEI, Eduqas)

Water pours into a cone, vertex down, at 10 cm^3s^{-1}. The surface radius is always $\frac{h}{3}$.

Find the rate at which the depth is rising when h = 10 cm.

$V = \frac{1}{3}\pi(h/3)^2h = \frac{\pi h^3}{27}$, so $\frac{dV}{dh} = \frac{\pi h^2}{9}$

$\frac{dh}{dt} = \frac{dV}{dt \div dV/dh} = \frac{10 \times 9}{\pi h^2}$

h = 10: $\frac{dh}{dt} = \frac{9}{10\pi}$ cm s^{-1}

Answer: $\frac{9}{10\pi}$ cm/s

8 Binomial expansion: (a + bx)^n for rational n (All boards)

Expand $(9 - 3x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including x^2.
State the range of x for which the expansion is valid.

$(9 - 3x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}(1 + \frac{-3x}{9})^{-\frac{1}{2}}$

$9^{-\frac{1}{2}} = \frac{1}{3}$

Expand $(1 + y)^{-\frac{1}{2}}$ with $y = \frac{-3x}{9}$

Valid for $|\frac{-3x}{9}| < 1$, i.e. $|x| < 3$

Answer: $\frac{1}{3} + \frac{1}{18}x + \frac{1}{72}x^2$; $|x| < 3$

9 Circle and line: chord length (AQA, Edexcel, OCR, MEI, Eduqas)

The line $-3x + 4y = 34$ meets the circle $(x + 3)^2 + (y + 2)^2 = 49$ at the points A and B. Find the exact length of AB.

Centre $(-3, -2)$, radius 7. Distance from centre to line:

$d = \frac{|-33|}{5} = \frac{33}{5}$

Half-chord$^2 = 49 - (\frac{33}{5})^2 = \frac{136}{25}$

$AB = 2\sqrt{\frac{136}{25}}$

Answer: $\frac{20\sqrt{34}}{25}$

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10 Closest approach of two particles (Edexcel, OCR, MEI)

At t = 0, P and Q have position vectors $(-10\mathbf{i} - \mathbf{j})$ m and $(-\mathbf{i} - 4\mathbf{j})$ m and velocities $(3\mathbf{i} + 2\mathbf{j})$ m s^{-1} and $(\mathbf{i} + \mathbf{j})$ m s^{-1}. Find (a) the time when they are closest, (b) the least distance between them. Give answers to 3 s.f. if not exact.

$PQ = r_P - r_Q = (-9\mathbf{i} + 3\mathbf{j}) + t(2\mathbf{i} + \mathbf{j})$

$d^2 = (2t - 9)^2 + (t + 3)^2 = 5t^2 - 30t + 90$

(a) $d(d^2)/dt = 0$: $10\,t + -30 = 0$, so $t = 3$

(b) At $t = 3$: $PQ = -3\mathbf{i} + 6\mathbf{j}$, so $d = \sqrt{45}$

Answer: (a) t = 3 s (b) 6.71 m