Day 263 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Stopping distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A driver at constant speed sees a hazard 62.25 m ahead. After a reaction time of 0.4 s she brakes with deceleration 2 m s^{-2} and stops exactly at the hazard. Find (a) her initial speed, (b) the total time from seeing the hazard to stopping.

Let the initial speed be v. Reaction distance = 0.4 v, braking distance = $v^2 \div (2\times2)$

$0.4\,v + v^2 \div 4 = 62.25$, so $v^2 + 1.6\,v - 249 = 0$

$(v - 15)(v + 16.6) = 0$, so $v = 15$

(b) Braking time = $15 \div 2 = 7.5$ s; total = 0.4 + 7.5

Answer: (a) 15 m s^{-1} (b) 7.9 s

2 Exact definite integrals of standard functions (All boards)

Find the exact value of

$\int_0^{\frac{\pi}{12}} 4\sec^2 3x\,dx$.

Integrate: $[\frac{4}{3}\tan 3x]$

Substitute the limits and subtract: $\frac{4}{3}$

Answer: $\frac{4}{3}$

3 Combined inequalities (All boards)

(a) Solve $5x + 6 < 3$.

(b) Solve $x^2 + 9x + 20 \geq 0$.

(c) Hence find the values of x satisfying both inequalities.

(a) $5x + 6 < 3$ so $x < -\frac{3}{5}$

(b) $(x + 5)(x + 4) \geq 0$: critical values -5 and -4

(b) gives $x \leq -5$ or $x \geq -4$

(c) Overlap of the two sets.

Answer: (a) $x < -\frac{3}{5}$ (b) $x \leq -5$ or $x \geq -4$ (c) $x \leq -5$ or $-4 \leq x < -\frac{3}{5}$

4 Newton's second law with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

A particle of mass 4 kg is acted on by forces $F_1 = (8\mathbf{i} - 7\mathbf{j})$ N and F_2. Its acceleration is $(3\mathbf{i} - \mathbf{j})$ m s^{-2}. Find (a) F_2, (b) the angle between F_2 and $\mathbf{i}$, to 1 d.p.

F = ma: $F_1 + F_2 = 4(3\mathbf{i} - \mathbf{j}) = 12\mathbf{i} - 4\mathbf{j}$

(a) $F_2 = (12\mathbf{i} - 4\mathbf{j}) - (8\mathbf{i} - 7\mathbf{j}) = 4\mathbf{i} + 3\mathbf{j}$

(b) $|F_2| = \sqrt{(4^2 + 3^2)} = 5$

$\cos\theta = 4 \div 5$, so $\theta = 36.9°$

Answer: (a) $\mathbf{F}_2 = (4\mathbf{i} + 3\mathbf{j})$ N (b) 36.9°

5 Hypothesis test for a normal mean (All boards)

The mean time taken (minutes) is normally distributed with standard deviation 5; it was previously 67.

A random sample of 9 has mean 68.9.

Test, at the 5% level, whether the mean has changed.

$H_0: \mu = 67$, $H_1: \mu \neq 67$

$\bar{X} \sim N(67, \frac{5^2}{9})$, $z = \frac{68.9 - 67}{\frac{5}{3}} = 1.14$

p-value = 0.254 (two-tailed) > 0.05

Do not reject H_0: no evidence that the mean has changed.

Answer: z = 1.14, p = 0.254; do not reject H_0

6 Product rule: stationary points (All boards)

$y = x^4 \ln x$ for $x > 0$.
Find the exact coordinates of the stationary point and show that it is a minimum.

$\frac{dy}{dx} = 4x^3 \ln x + x^4 \times \frac{1}{x} = x^3(4 \ln x + 1)$

$\frac{dy}{dx} = 0$, $x > 0$: $\ln x = -\frac{1}{4}$, so $x = e^{-\frac{1}{4}}$

$y = x^4 \ln x = e^{-1} \times (-\frac{1}{4}) = -\frac{e^{-1}}{4}$

$\frac{d^2y}{dx^2} = 4e^{-\frac{1}{2}} > 0$ at this point, so it is a minimum

Answer: $(e^{-\frac{1}{4}}, -\frac{e^{-1}}{4})$, minimum

7 Finding an unknown standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X is normally distributed with mean μ and standard deviation 6.
Given that $P(X > 55) = 0.1587$, find μ to 3 significant figures.

$P(Z > z) = 0.1587$, so $z = 1.00$

$\frac{55 - \mu}{6} = 1.00$

$\mu = 55 - (1.00)(6) = 49.0$

Answer: $\mu = 49.0$

8 Binomial expansion: (a + bx)^n for rational n (All boards)

Expand $(27 - 3x)^{\frac{1}{3}}$ in ascending powers of x, up to and including x^2.
State the range of x for which the expansion is valid.

$(27 - 3x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1 + \frac{-3x}{27})^{\frac{1}{3}}$

$27^{\frac{1}{3}} = 3$

Expand $(1 + y)^{\frac{1}{3}}$ with $y = \frac{-3x}{27}$

Valid for $|\frac{-3x}{27}| < 1$, i.e. $|x| < 9$

Answer: $3 - \frac{1}{9}x - \frac{1}{243}x^2$; $|x| < 9$

9 Arithmetic modelling (All boards)

A charity raises £600 in its first year and in each later year raises £40 more than in the previous year.
(a) How much does it raise in year 11? (b) In which year does the total raised first exceed £5000?

(a) $u_{11} = 600 + 10 \times 40 = 1000$

(b) $S_m = \frac{m}{2}(1200 + (m - 1) \times 40) > 5000$

$S_6 = 4200 < 5000$ and $S_7 = 5040 > 5000$

Answer: (a) £1000 (b) year 7

10 Equations with cosec x and cot x (All boards)

Solve $3\text{cosec}^2 x - 9\cot x - 15 = 0$
for $0° < x < 360°$. Give answers to 1 d.p.

Use $\text{cosec}^2 x \equiv 1 + \cot^2 x$: $3\cot^2 x - 9\cot x - 12 = 0$
$\cot x = -1$ or $\cot x = 4$
$\cot x = -1$ means $\tan x = -1$: $x = 135.0°, 315.0°$
$\cot x = 4$ means $\tan x = \frac{1}{4}$: $x = 14.0°, 194.0°$

Answer: $x = 14.0°, 135.0°, 194.0°, 315.0°$