Day 262 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Circle with a given diameter (AQA, Edexcel, OCR, MEI, Eduqas)

A(7, 7) and B(1, −1) are the ends of a diameter of a circle.
(a) Find the equation of the circle, in the form $x^2 + y^2 + ax + by + c = 0$.
(b) Show that C(4, 8) lies on the circle.

Centre = midpoint of AB = (4, 3)
$r^2 = (3)^2 + (4)^2 = 25$
$(x - 4)^2 + (y - 3)^2 = 25$: $x^2 + y^2 - 8x - 6y = 0$
(b) $CM^2 = (0)^2 + (5)^2 = 25 = r^2$
Answer: (a) $x^2 + y^2 - 8x - 6y = 0$

2 Car and trailer (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 1500 kg car tows a 300 kg trailer by a light tow-bar on a horizontal road. Driving force 1060 N; resistances 500 N (car), 200 N (trailer). Find (a) the acceleration and the tow-bar tension. At 12 m s^{-1} the tow-bar breaks. (b) How far does the trailer then travel?

Whole system: 1060 − 500 − 200 = 1800 a, so a = 0.2
Trailer: T − 200 = 300 a, so T = 200 + 300×0.2 = 260
(b) After the break the trailer decelerates at 200 ÷ 300 = $\frac{2}{3}$ m s^{-2}
$0 = 12^2 - 2\times\frac{2}{3}\times s$, so s = 108
Answer: (a) 0.2 m s^{-2}, 260 N (b) 108 m

3 Geometric distribution (CIE)

On each independent trial, the probability that a bulb is faulty is $\frac{1}{3}$.
X is the number of trials up to and including the first success.
(a) Find P(X = 5) and E(X).
(b) Find the least m with P(X ≤ m) > 0.95.

(a) $P(X = 5) = (\frac{2}{3})^4 \times \frac{1}{3} = \frac{16}{243}$; $E(X) = \frac{1}{p} = 3$
(b) $P(X \leq m) = 1 - (\frac{2}{3})^m > 0.95$
$(\frac{2}{3})^m < 0.05$, so m > 7.39
Least integer m = 8
Answer: (a) $\frac{16}{243}$, E(X) = 3 (b) m = 8

4 Ships on collision courses (vectors) (AQA, Edexcel, OCR, MEI, Eduqas)

At 12:00 ship P has position (13**i**) km and velocity (−3**i** − 3**j**) km h^{-1}; ship Q has position (**i** − 18**j**) km and velocity (3**i** + 6**j**) km h^{-1}. Show that the ships collide and find the time and position vector of the collision.

At time t hours: $\mathbf{r}_P = (13\mathbf{i}) + t(-3\mathbf{i} - 3\mathbf{j})$, $\mathbf{r}_Q = (\mathbf{i} - 18\mathbf{j}) + t(3\mathbf{i} + 6\mathbf{j})$
Equal when: **i**: 13 + −3 t = 1 + 3 t, so t = 2
j: 0 + 2×(−3) = −6
Check with Q: **i**: 1 + 2×3 = 7, **j**: −18 + 2×6 = −6
Both components agree at t = 2, so they collide at (7**i** − 6**j**) km
Answer: t = 2 hours (at 14:00); r = (7i − 6j) km

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5 Implicit differentiation: tangent to a curve (All boards)

A curve has equation $3x^2 + 2xy + 2y = -1$.
Show that P(1, −1) lies on the curve and find the equation of the tangent at P in the form ax + by + c = 0 with integer a, b, c.

Differentiate (product rule on the xy term): $(6x + 2y) + (2x + 2)\frac{dy}{dx} = 0$

At P(1, −1): $4 + (4)\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -1$

Tangent: $y + 1 = (-1)(x - 1)$, i.e. $x + y = 0$

Answer: $x + y = 0$

6 Finding mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, \sigma^2)$, with $P(X < 48) = 0.3085$ and $P(X > 57) = 0.1587$.
Find μ and σ.

z-values: 48: $z = -0.500$; 57: $z = 1.00$
$48 = \mu - 0.500\,\sigma$ and $57 = \mu + 1.00\,\sigma$
Subtract: $9 = 1.50\,\sigma$, so $\sigma = 6.00$
$\mu = 48 + 0.500 \times 6.00 = 51.0$

Answer: $\mu = 51$, $\sigma = 6$

7 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

The radius of a sphere increases at $\frac{3}{2}$ cm s^{-1}.

Find the rate of increase of its volume when r = 2 cm, in terms of π.

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi \times 2^2 \times \frac{3}{2} = 24\pi$

Answer: 24π cm^3/s

8 Range of a composite function (All boards)

$f(x) = x^2 + 4x + 6$, $x \in \mathbb{R}$, and $g(x) = \frac{8}{x + 1}$, $x > -1$.

(a) Find the range of f.
(b) Show that gf is defined and find its range.

$f(x) = (x + 2)^2 + 2 \geq 2$
So $f(x) \geq 2 > -1$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{8}{f(x) + 1}$, with $f(x) \geq 2$, so $f(x) + 1 \geq 3$

Largest value is $\frac{8}{3} = \frac{8}{3}$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 2$ (b) $0 < gf(x) \leq \frac{8}{3}$

9 Binomial expansion: approximating a square root (All boards)

(a) Expand $(1 + 2x)^{\frac{1}{2}}$ in ascending powers of x up to x^2.
(b) Use x = 0.01 to estimate $\sqrt{102}$ to 4 decimal places.

$(1 + y)^{\frac{1}{2}} = 1 + \frac{1}{2}y - \frac{1}{8}y^2$, $y = 2x$

(a) $1 + x - \frac{1}{2}x^2$

$x = \frac{1}{100}$: $102 = 10^2(1 + \frac{1}{50})$ so $\sqrt{102} = 10 \times (1 + \frac{1}{50})^{\frac{1}{2}}$

$\approx 10 \times (1 + \frac{1}{100} - \frac{1}{20000}) = 10.099500$

Answer: (a) $1 + x - \frac{1}{2}x^2$ (b) 10.0995

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10 Quotient rule: stationary points (All boards)

Find the exact coordinates of the stationary point of

$y = \frac{e^{3x}}{x}$ for $x > 0$.

$\frac{dy}{dx} = \frac{x \times 3e^{3x} - e^{3x} \times 1}{x^2} = \frac{e^{3x}(3x - 1)}{x^2}$

$e^{3x} > 0$, so $\frac{dy}{dx} = 0$ when $3x = 1$, giving $x = \frac{1}{3}$

$y = e^1 \div \frac{1}{3} = 3e$

Answer: $\left(\frac{1}{3}, 3e\right)$