Day 25 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Surds: sums of fractions (AQA, Edexcel, OCR, MEI, Eduqas)

Show that $\frac{2}{\sqrt{3}-1} + \frac{5}{\sqrt{3}+1}$ can be written as $p + q\sqrt{3}$, and find p and q.

Common denominator $(\sqrt{3}-1)(\sqrt{3}+1) = 3 - 1 = 2$

Numerator $= 2(\sqrt{3}+1) + 5(\sqrt{3}-1)$

$= -3 + 7\sqrt{3}$

Divide by 2.

Answer: $p = -\frac{3}{2}$, $q = \frac{7}{2}$

2 Change of sign to locate a root (All boards)

$f(x) = \ln x + x - 7$.

Show that the equation $f(x) = 0$ has a root α in the interval $[5, 6]$.

f(x) is continuous on the interval

$f(5) = -0.391 < 0$

$f(6) = 0.792 > 0$

A change of sign, so a root lies in [5, 6]

Answer: Sign change: $f(5) = -0.391$, $f(6) = 0.792$

3 Combined inequalities (All boards)

(a) Solve $4x + 6 \leq 6$.

(b) Solve $x^2 + x - 20 < 0$.

(c) Hence find the values of x satisfying both inequalities.

(a) $4x + 6 \leq 6$ so $x \leq 0$

(b) $(x + 5)(x - 4) < 0$: critical values -5 and 4

(b) gives $-5 < x < 4$

(c) Overlap of the two sets.

Answer: (a) $x \leq 0$ (b) $-5 < x < 4$ (c) $-5 < x \leq 0$

4 Geometric distribution (CIE)

On each independent trial, the probability that a basketball player scores is $\frac{1}{6}$.

X is the number of trials up to and including the first success.

(a) Find P(X = 6) and E(X).

(b) Find the least m with $P(X \leq m) > 0.9$.

(a) $P(X = 6) = (\frac{5}{6})^5 \times \frac{1}{6} = \frac{3125}{46656}$; $E(X) = \frac{1}{p} = 6$

(b) $P(X \leq m) = 1 - (\frac{5}{6})^m > 0.9$

$(\frac{5}{6})^m < 0.1$, so $m > 12.6$

Least integer $m = 13$

Answer: (a) $\frac{3125}{46656}$, $E(X) = 6$ (b) $m = 13$

5 Circle and line: tangent condition (AQA, Edexcel, OCR, MEI, Eduqas)

The line $5x + 12y = k$ is a tangent to the circle

$(x + 2)^2 + (y - 4)^2 = 36$.

Find the two possible values of k.

Centre $(-2, 4)$, radius 6

Distance from centre to the line = radius:

$\frac{|38 - k|}{13} = 6$

$|38 - k| = 78$

Answer: $k = -40$ or $k = 116$

6 Projected up a rough incline (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 10.5 m s^{-1} up a line of greatest slope of a rough plane inclined at α, where $\tan\alpha = \frac{12}{5}$. The coefficient of friction is 0.1. Find (a) the greatest distance it moves up the plane, (b) its speed on returning to the start. Take g = 9.8. Give answers to 3 s.f. if not exact.

Up: deceleration = $g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{12}{13} + 0.1\times\frac{5}{13}) = \frac{245}{26}$

(a) $0 = 10.5^2 - 2\times\frac{245}{26}\times d$, so d = 5.85

Down: $\tan\alpha = \frac{12}{5} > \mu$, so it slides back; acceleration = $g(\sin\alpha - \mu\cos\alpha) = \frac{1127}{130}$

(b) $v^2 = 2\times\frac{1127}{130}\times 5.85 = 101.43$

$v = \sqrt{101.43}$

Answer: (a) 5.85 m (b) 10.1 m s^{-1}

7 Greatest and least values using R-form (All boards)

$g(\theta) = \frac{1}{19 + 8\sin\theta + 15\cos\theta}$

(a) Write $8\sin\theta + 15\cos\theta$ in the form $R\sin(\theta + \alpha)$, with α acute.

(b) Find the greatest and least values of $g(\theta)$ and the smallest positive θ at which the greatest value occurs (1 d.p.).

$R = \sqrt{8^2 + 15^2} = 17$, $\alpha = \tan^{-1}\frac{15}{8} = 61.9°$

Denominator = $19 + 17\sin(\theta + 61.9°)$, ranging from 2 to 36

Greatest g = $\frac{1}{2}$ (denominator least), least g = $\frac{1}{36}$

Least denominator when $\sin(\theta + 61.9°) = -1$: $\theta + 61.9° = 270°$

$\theta = 208.1°$

Answer: R = 17, α = 61.9°; greatest $\frac{1}{2}$, least $\frac{1}{36}$; θ = 208.1°

8 Conditional probability and independence (All boards)

P(B) = 0.35, P(A) = 0.65 and P(A | B) = 0.15.

(a) Find $P(A \cap B)$ and $P(A \cup B)$.

(b) Find P(B | A).

(c) Determine whether A and B are independent.

(a) $P(A \cap B) = P(A \mid B)P(B) = 0.15 \times 0.35 = \frac{21}{400}$

$P(A \cup B) = 0.65 + 0.35 - \frac{21}{400} = \frac{379}{400}$

(b) $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{21}{400}}{0.65} = \frac{21}{260}$

(c) $P(A)P(B) = 0.65 \times 0.35 = \frac{91}{400}$, which does not equal $P(A \cap B) = \frac{21}{400}$

So A and B are not independent

Answer: (a) $\frac{21}{400}$, $\frac{379}{400}$ (b) $\frac{21}{260}$ (c) not independent

9 Reducing a relationship to a straight line (graph) (All boards)

The diagram shows the graph of ln y against x for $y = A e^{kx}$.

Find the exact values of A and k.

$\ln y = \ln A + kx$, a straight line with gradient k and intercept ln A

Intercept $\ln A = 1$, so $A = e^1$

Gradient $k = \frac{4 - 1}{2 - 0} = \frac{3}{2}$

Answer: A = e^1, k = $\frac{3}{2}$

10 Position vector and velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P has position vector $\mathbf{r} = (2t^3 + 3t)\mathbf{i} + (t^2 - 6t)\mathbf{j}$ metres at time t s. Find (a) the speed of P when t = 2, (b) the value of t when P is moving parallel to $\mathbf{i}$. Give answers to 3 s.f. if not exact.

$\mathbf{v} = d\mathbf{r}/dt = (6t^2 + 3)\mathbf{i} + (2t - 6)\mathbf{j}$

(a) $t = 2$: $\mathbf{v} = 27\mathbf{i} - 2\mathbf{j}$

Speed $= \sqrt{(27^2 + -2^2)} = \sqrt{733}$

(b) Parallel to $\mathbf{i}$ means the $\mathbf{j}$ component is zero: $2t - 6 = 0$, so $t = 3$

Answer: (a) 27.1 m s^{-1} (b) t = 3