Day 249 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Correcting a recorded value (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

10 values have $\Sigma x = 350$ and $\Sigma x^2 = 12500$.
One value was recorded as 48 but should have been 38.
Find the correct mean and standard deviation (3 s.f.).

$\Sigma x = 350 - 48 + 38 = 340$

$\Sigma x^2 = 12500 - 48^2 + 38^2 = 11640$

Mean $= \frac{340}{10} = 34.0$

sd $= \sqrt{\frac{11640}{10} - (34.0)^2} = 2.83$

Answer: mean 34.0, sd 2.83

2 Exact values from the addition formulae (All boards)

By writing 75° as 45° + 30°, show that
tan 75° = $\sqrt{3} + 2$.

$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$\tan 30° = \frac{\sqrt{3}}{3}$, $\tan 45° = 1$, $\tan 60° = \sqrt{3}$, $\tan 120° = -\sqrt{3}$, $\tan 150° = -\frac{\sqrt{3}}{3}$

$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \times \frac{\sqrt{3}}{3}}$, which rationalises to $\sqrt{3} + 2$

Answer: $\sqrt{3} + 2$

3 Conservation of momentum (CIE)

Particles A (8 kg) and B (5 kg) move in the same direction on a smooth horizontal line at 5 m s^{-1} and 5 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):
$8 \times 5 + 5 \times 5 = 13v$, so $v = 5$
KE before $= \frac{1}{2} \times 8 \times 5^2 + \frac{1}{2} \times 5 \times 5^2 = 162.5$
KE after $= \frac{1}{2} \times 13 \times 5^2 = 162.5$
Loss $= 162.5 - 162.5$

Answer: (a) 5 m s^{-1} (b) 0 J

4 Quadratic from its graph (All boards)

The diagram shows the curve with equation y = f(x), where f is a quadratic.
The turning point is marked, and the curve crosses the y-axis as shown.
Find f(x), giving it in the form $ax^2 + bx + c$.

Turning point (4, 7): $f(x) = a(x - 4)^2 + 7$
$f(0) = 23$: $a(4)^2 + 7 = 23$
$a = 1$, then expand

Answer: $f(x) = x^2 - 8x + 23$

5 Addition formulae with given sin and cos (All boards)

A is acute and B is obtuse, with $\sin A = \frac{3}{5}$ and $\cos B = -\frac{15}{17}$.

Find the exact values of sin(A+B) and cos(A+B).

$\cos A = \frac{4}{5}$ (A acute), $\sin B = \frac{8}{17}$ (B obtuse), so $\tan A = \frac{3}{4}$, $\tan B = -\frac{8}{15}$

$\sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \times (-\frac{15}{17}) + \frac{4}{5} \times \frac{8}{17} = -\frac{13}{85}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times (-\frac{15}{17}) - \frac{3}{5} \times \frac{8}{17} = -\frac{84}{85}$

Answer: $\sin(A+B) = -\frac{13}{85}$, $\cos(A+B) = -\frac{84}{85}$

6 Foot of a perpendicular (All boards)

The line l has equation $2x + y = -4$.
(a) Find the coordinates of the foot of the perpendicular from P$(-5, 5)$ to l.
(b) Hence find the shortest distance from P to l, in exact form.

Gradient of l $= -2$, so perpendicular gradient $= \frac{1}{2}$

Perpendicular through P: $x - 2y = -15$

Solve with l: $x = -\frac{23}{5}$, $y = \frac{26}{5}$

Distance$^2 = (-5 - (-\frac{23}{5}))^2 + (5 - \frac{26}{5})^2 = \frac{1}{5}$

Answer: (a) $(-\frac{23}{5}, \frac{26}{5})$ (b) $\frac{\sqrt{5}}{5}$

7 Symmetric interval about the mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the journey time (minutes), is N(39, 6^2).
Find the symmetric limits about the mean between which the central 95% of values lie (3 s.f.).

Each tail has area 0.025, so P(Z < z) = 0.975
z = 1.96
Limits: $39 \pm 1.96 \times 6$
= 27.2 and 50.8
Answer: 27.2 to 50.8

8 Tangent through the origin (All boards)

The tangent to the curve $y = e^{\frac{1}{2}x}$ at the point P passes through the origin.
Find the coordinates of P and the equation of the tangent.

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$; at x = a the tangent is $y - e^{\frac{1}{2}a} = \frac{1}{2}e^{\frac{1}{2}a}(x - a)$

Through (0, 0): $-e^{\frac{1}{2}a} = -\frac{1}{2}a\, e^{\frac{1}{2}a}$, so $\frac{1}{2}a = 1$ and $a = 2$

P = (2, e) and gradient ke $= \frac{1}{2}e$

Answer: P(2, e); $y = \frac{1}{2}ex$

9 Recurrence relations: limits (All boards)

$u_1 = -1$ and $u_{n+1} = \frac{1}{3}u_n + 6$.
(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.001$.

(a) $L = \frac{1}{3}L + 6$ gives $L = \frac{6}{1 - \frac{1}{3}} = 9$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = (-10) \times \frac{1}{3}^{n-1}$

$|-10| \times \frac{1}{3}^{n-1} < \frac{1}{1000}$

Check: n = 9 gives 0.00152, n = 10 gives 0.00051
Answer: (a) L = 9 (b) n = 10

10 Projectile from a cliff top (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a cliff top 62.5 m above the sea at 20 m s^{-1}, at angle α above the horizontal, $\tan\alpha = \frac{3}{4}$. Find (a) the time to reach the sea, (b) the horizontal distance travelled, (c) the speed on impact. Take g = 9.8. Give answers to 3 s.f. if not exact.

Horizontal component 16, vertical component 12 (upwards)
(a) $-62.5 = 12\,t - 4.9t^2$, so $4.9t^2 - 12\,t - 62.5 = 0$
$t = [12 + \sqrt{1369}] \div 9.8 = 5$ (taking the positive root)
(b) Distance = 16×5
(c) Vertical speed = $12 - 9.8\times5 = -37$; speed = $\sqrt{(16^2 + 37^2)}$
Answer: (a) 5 s (b) 80 m (c) 40.3 m s^{-1}