Day 24 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Implicit differentiation: products and cubes (All boards)

A curve has equation $3x^2y - y^3 - 3x = -5$.

Find $\frac{dy}{dx}$ in terms of x and y, and the gradient of the normal at $(-2, -1)$.

Differentiate: $6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 3 = 0$

$\frac{dy}{dx} = -\frac{6xy - 3}{3x^2 - 3y^2}$

At $(-2, -1)$: $\frac{dy}{dx} = -1$, so the normal has gradient 1

Answer: $\frac{dy}{dx} = -\frac{6xy - 3}{3x^2 - 3y^2}$; normal gradient 1

2 One-tailed binomial hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

A claim is that the probability p that a seed germinates is 0.2. In a sample of 18, the number of successes is 6. Test at the 5% level whether p has increased.

H_0: p = 0.2, H_1: p > 0.2; X ~ B(18, 0.2)

$P(X \geq 6) = 1 - P(X \leq 5) = 0.133$

0.133 > 0.05

Do not reject H_0: insufficient evidence of a change in p.

Answer: p-value 0.133; do not reject H_0

3 Differentiating a^x (All boards)

$y = 5 \times 3^x$.

Find the exact gradient of the curve at x = 2 and the equation of the tangent there.

$\frac{dy}{dx} = 5 \times 1 \ln 3 \times 3^x = 5 \ln 3 \times 3^x$

At x = 2: y = 45 and gradient = 45 ln 3

Tangent: $y = (45 \ln 3)x + 45 - 90 \ln 3$

Answer: gradient 45 ln 3; $y = (45 \ln 3)x + 45 - 90 \ln 3$

4 Partial fractions: repeated factor (All boards)

Express $\frac{10x^2 + 2x - 16}{(x - 3)(x + 1)^2}$ in the form $\frac{A}{(x - 3)} + \frac{B}{(x + 1)} + \frac{C}{(x + 1)^2}$, and find A, B and C.

$10x^2 + 2x - 16 \equiv A(x + 1)^2 + B(x - 3)(x + 1) + C(x - 3)$

x = 3: $80 = A(4)^2$ so A = 5

x = −1: $-8 = C(-4)$ so C = 2

Coefficient of x^2: 10 = A + B so B = 5

Answer: A = 5, B = 5, C = 2

5 Non-uniform rod and equal tensions (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB (length 4 m, mass 8 kg) has its centre of mass at G, AG = 0.75 m. It hangs horizontally from vertical strings at A and B. A particle of mass m kg is fixed at P, AP = 2.5 m, making the tensions equal. Find (a) m, (b) the tension. Take g = 9.8.

Equal tensions T: 2T = (8 + m)g, so T = ½(8 + m)g

Moments about A: T×4 = 8 g×0.75 + m g×2.5

½(8 + m)×4 = 6 + 2.5 m → 16 + 2 m = 6 + 2.5 m

m (2.5 − 2) = 16 − 6, so m = 20

(b) T = ½(8 + 20)×9.8

Answer: (a) m = 20 (b) 137.2 N

6 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

The volume of a cube of edge x cm is increasing at 18 cm^3s^{-1}.
Find the rate of increase of its total surface area when x = 3.

$V = x^3$, $\frac{dV}{dx} = 3x^2$, so $\frac{dx}{dt} = \frac{18}{3x^2} = \frac{2}{3}$ at x = 3

$S = 6x^2$, $\frac{dS}{dx} = 12x$, so $\frac{dS}{dt} = 12 \times 3 \times \frac{2}{3} = 24$

Answer: 24 cm^2/s

7 Modulus inequalities (All boards)

Solve $|4x + 6| > |2x - 6|$.

Square both sides: $(4x + 6)^2 > (2x - 6)^2$
$((4x + 6) - (2x - 6))((4x + 6) + (2x - 6)) > 0$
$(2x + 12)(6x) > 0$
Critical values $x = -6$ and $x = 0$; sketch the (positive or negative) quadratic.
Answer: $x < -6$ or $x > 0$

8 Passing a point twice (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is projected vertically upwards from the ground. It passes a ledge 4 s after projection and again 6 s after projection. Find (a) the speed of projection, (b) the height of the ledge, (c) the greatest height. Take g = 9.8.

Same height at t = 4 and t = 6, so the stone is at its highest point at t = 5
(a) At the top v = 0: $0 = u - 9.8 \times 5$, so u = 49
(b) $s = ut - 4.9t^2$ with t = 4: $s = 49 \times 4 - 4.9 \times 4^2 = 117.6$
(c) $v^2 = u^2 - 2gs$: $0 = 49^2 - 19.6s$, s = 122.5
Answer: (a) 49 m s^{-1} (b) 117.6 m (c) 122.5 m

9 Sum to infinity: two possible ratios (All boards)

The second term of a geometric series is 2 and its sum to infinity is 9.
Find the two possible values of the first term.

$ar = 2$ and $\frac{a}{(1 - r)} = 9$
$a = 9(1 - r)$, so $9(1 - r)\, r = 2$
$9r^2 - 9r + 2 = 0$: $r = \frac{1}{3}$ or $r = \frac{2}{3}$
$a = 9(1 - r) = 6$ or 3
Answer: a = 3 or a = 6

10 Two-tailed binomial test critical region (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(22, p)$. H_0: p = 0.25 is tested against H_1: $p \neq 0.25$ at the 5% level.
(a) Find the critical region.
(b) Find the actual significance level of the test (3 s.f.).

(a) Each tail ≤ 0.025. $P(X \leq 1) = 0.0149$, $P(X \leq 2) = 0.0606$
$P(X \geq 11) = 0.00997$, $P(X \geq 10) = 0.0295$
Critical region: $X \leq 1$ or $X \geq 11$
(b) Actual level = 0.0149 + 0.00997 = 0.0248
Answer: (a) $X \leq 1$ or $X \geq 11$ (b) 0.0248