Day 223 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Transformations from a graph (All boards)

The diagram shows the curve y = f(x), with turning points A and B.
Find the coordinates of the turning points of
(a) y = f(x − 2) (b) y = −f(x)

(a) y = f(x − 2): A → (1, −6), B → (3, 2)
(b) y = −f(x): A → (−1, 6), B → (1, −2)
Answer: (a) (1, −6), (3, 2); (b) (−1, 6), (1, −2)

2 Integration by substitution (All boards)

Use the substitution u = ln x to find the exact value of
$\int_1^e \frac{(\ln x)^3}{x}\,dx$.

$du = \frac{1}{x}dx$; x = 1 gives u = 0, x = e gives u = 1

$I = \int_0^1 u^3\,du = [\frac{u^4}{4}]$

$= \frac{1}{4}$

Answer: $\frac{1}{4}$

3 Independent and mutually exclusive events (All boards)

P(A) = 0.6 and P(A ∪ B) = 0.8.
(a) Given that A and B are independent, find P(B).
(b) Given instead that A and B are mutually exclusive, find P(B).
(c) In the case in (a), find P(A′ ∩ B).

(a) P(A ∪ B) = P(A) + P(B) − P(A)P(B), so 0.8 = 0.6 + P(B)(1 − 0.6)
$P(B) = \frac{0.8 - 0.6}{0.4} = 0.5$
(b) P(A ∩ B) = 0, so P(B) = 0.8 − 0.6 = 0.2
(c) P(A′ ∩ B) = P(A′)P(B) = 0.4 × 0.5 = 0.2
Answer: (a) 0.5 (b) 0.2 (c) 0.2

4 Implicit differentiation: tangent to a curve (All boards)

A curve has equation $-x^2 + 4xy - 3y^2 + y = -10$.
Show that P(−2, −3) lies on the curve and find the equation of the tangent at P in the form ax + by + c = 0 with integer a, b, c.

Differentiate (product rule on the xy term): $(-2x + 4y) + (4x - 6y + 1)\frac{dy}{dx} = 0$

At P(−2, −3): $-8 + (11)\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{8}{11}$

Tangent: $y + 3 = \frac{8}{11}(x + 2)$, i.e. 8x − 11y − 17 = 0

Answer: 8x − 11y − 17 = 0

5 Foot of a perpendicular (All boards)

The line l has equation 2x − y = 4.
(a) Find the coordinates of the foot of the perpendicular from P(−4, 1) to l.
(b) Hence find the shortest distance from P to l, in exact form.

Gradient of l = 2, so perpendicular gradient = $-\frac{1}{2}$

Perpendicular through P: −x − 2y = 2

Solve with l: $x = \frac{6}{5}$, $y = -\frac{8}{5}$

$\text{Distance}^2 = (-4 - \frac{6}{5})^2 + (1 - (-\frac{8}{5}))^2 = \frac{169}{5}$

Answer: (a) $(\frac{6}{5}, -\frac{8}{5})$ (b) $\frac{13\sqrt{5}}{5}$

6 Pulley, string goes slack (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (3 kg) and B (1 kg) hang on a light string over a smooth pulley, A being 0.5 m above the ground. Released from rest; A lands without rebounding. Find (a) the tension while both move, (b) the greatest height of B above its start. Take g = 9.8.

A: $3g - T = 3a$; B: $T - 1g = 1a$
$a = 2g \div 4 = 4.9$, so $T = 1(g + a) = 14.7$
When A lands, B has risen 0.5 m with $v^2 = 2 \times 4.9 \times 0.5 = 4.9$
B then moves freely under gravity: extra height $= v^2 \div (2 \times 9.8) = 0.25$
Total = 0.5 + 0.25
Answer: (a) 14.7 N (b) 0.75 m

7 Exponential equations reducing to quadratics (All boards)

Solve $3^{2x} - 26 \cdot 3^x + 153 = 0$.
Give exact roots, or 3 s.f. where the answer is not exact.

Let $u = 3^x$: $u^2 - 26u + 153 = 0$, so $(u - 9)(u - 17) = 0$
$3^x = 9$ gives $x = 2$; $3^x = 17$ gives $x = \frac{\ln 17}{\ln 3} = 2.58$
Answer: x = 2 or x = 2.58

8 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The length of a plank is $N(50, 3^2)$. A plank is classed as 'special' if its length is more than 53.
12 planks are chosen at random. Find the probability that exactly 3 are special (3 s.f.).

P(special) = P(X > 53) = P(Z > 1.00) = 0.159
$Y \sim B(12, 0.159)$
$P(Y = 3) = C(12, 3)(0.159)^3(0.841)^9 = 0.186$
Answer: 0.186

9 Binomial expansion: approximating a square root (All boards)

(a) Expand $(1 - x)^{\frac{1}{2}}$ in ascending powers of x up to x^2.
(b) Use x = 0.01 to estimate $\sqrt{99}$ to 3 decimal places.

$(1 + y)^{\frac{1}{2}} = 1 + \frac{1}{2}y - \frac{1}{8}y^2$, $y = -1x$
(a) $1 - \frac{1}{2}x - \frac{1}{8}x^2$
$x = \frac{1}{100}$: $99 = 10^2(1 - \frac{1}{100})$ so $\sqrt{99} = 10 \times (1 - \frac{1}{100})^{\frac{1}{2}}$
$\approx 10 \times (1 - \frac{1}{200} - \frac{1}{80000}) = 9.949875$
Answer: (a) $1 - \frac{1}{2}x - \frac{1}{8}x^2$ (b) 9.950

10 Particles on two smooth inclined planes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The system is released from rest. Find (a) the magnitude of the acceleration and which particle moves down, (b) the tension. Take g = 9.8. Give answers to 3 s.f. if not exact.

A: weight component along plane = $8g \sin 30°$; B: $9g \sin 30°$
B moves down, A up ($9 \sin 30° > 8 \sin 30°$): F = ma for the system
$9g \sin 30° - 8g \sin 30° = (8 + 9)a$
(a) a = 0.288
(b) For A: $T - 8g \sin 30° = 8a$, so T = 41.5
Answer: (a) 0.288 m s^{-2}, B moves down (b) 41.5 N