Day 180 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Factor theorem: roots with surds (All boards)

$f(x) = x^3 - 8x^2 + 20x - 15$.
(a) Show that $(x - 3)$ is a factor of $f(x)$.
(b) Solve $f(x) = 0$, giving surds in simplest form.

$f(3) = 27 - 72 + 60 - 15 = 0$, so $(x - 3)$ is a factor.

$f(x) = (x - 3)(x^2 - 5x + 5)$

$x^2 - 5x + 5 = 0$ gives $x = \frac{5 \pm \sqrt{5}}{2}$

Answer: $x = 3$ or $x = \frac{5 \pm \sqrt{5}}{2}$

2 Stratified sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A stratified sample of 80 students is taken from three groups.
9 of the sample are from Arts, which has 54 students.
(a) Find the total number of students.
(b) Science has 78 students. How many are sampled?

Arts sampled fraction $= \frac{9}{54} = \frac{1}{6}$

Total $= 80 \div \frac{1}{6} = 480$

Science: $78 \times \frac{1}{6} = 13$

Answer: (a) 480 (b) 13

3 Equations quadratic in disguise (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^4 - 26x^2 + 25 = 0$.

Let $u = x^2$: $u^2 - 26u + 25 = 0$
$(u - 25)(u - 1) = 0$ so $u = 25$ or $u = 1$
$x^2 = 25$ gives $x = \pm 5$
$x^2 = 1$ gives $x = \pm 1$
Answer: $x = \pm 1, \pm 5$

4 Newton's second law with a time-varying force (AQA, Edexcel, OCR, MEI)

A 8 kg particle moves on a smooth horizontal line. At time t s a single force of $(16 - 4t)$ N acts along the line. When $t = 0$ its speed is 0 m s^{-1}. Find (a) its speed when $t = 5$, (b) its greatest speed.

$F = ma$: $a = (-4t + 16) \div 8$
$v = \int a\,dt = 2t - 0.25t^2 + c$; $v = 0$ when $t = 0$
(a) $v(5) = 3.75$
(b) Greatest speed when $a = 0$: $t = 4$
$v(4) = 4$
Answer: (a) 3.75 m s^{-1} (b) 4 m s^{-1}

5 Newton-Raphson method (AQA, Edexcel, OCR, MEI, Eduqas)

$f(x) = e^x + 1x - 8$.
Use the Newton-Raphson method with $x_0 = 2$ to find x_1 and x_2, giving your answers to 4 d.p.

$f'(x) = e^x + 1$

$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1.3891}{8.3891} = 1.8344$

$x_2 = 1.8344 - \frac{f(1.8344)}{f'(1.8344)} = 1.8212$

Answer: $x_1 = 1.8344$, $x_2 = 1.8212$

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6 Maximum speed from velocity function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle moves in a straight line with velocity $v = (-6t^2 + 24t)$ m s^{-1} at time t s, $t \geq 0$, starting at O. Find (a) its greatest speed, (b) its distance from O when t = 4, (c) the time at which it returns to O.

(a) $dv/dt = 24 - 12t = 0$, so $t = 2$ (a maximum since the t^2 coefficient is negative)
$v = 24 \times 2 - 6 \times 2^2 = 24$
(b) $x = \int v\, dt = 12\, t^2 - 2\, t^3$
$x(4) = 64$
(c) $x = 0$ when $t^2(12 - 2\, t) = 0$, so $t = 6$

Answer: (a) 24 m s^{-1} (b) 64 m (c) 6 s

7 Continuous random variable: variance (CIE)

X has pdf $f(x) = \frac{6}{27}x(3 - x)$ for $0 \leq x \leq 3$, and 0 otherwise.

Find E(X), Var(X) and the standard deviation of X (3 s.f.).

$E(X) = \int_0^3 x\, f(x)\, dx = \frac{3}{2}$

$E(X^2) = \int_0^3 x^2\, f(x)\, dx = \frac{27}{10}$

$\text{Var}(X) = \frac{27}{10} - (\frac{3}{2})^2 = \frac{9}{20}$

Standard deviation $= \sqrt{\frac{9}{20}} = 0.671$

Answer: E(X) = $\frac{3}{2}$, Var(X) = $\frac{9}{20}$, sd 0.671

8 Exponential equations reducing to quadratics (All boards)

Solve $2^{2x} - 202^x + 36 = 0$.
Give exact roots, or 3 s.f. where the answer is not exact.

Let $u = 2^x$: $u^2 - 20\, u + 36 = 0$, so $(u - 2)(u - 18) = 0$

$2^x = 2$ gives $x = 1$; $2^x = 18$ gives $x = \frac{\ln 18}{\ln 2} = 4.17$

Answer: x = 1 or x = 4.17

9 Range of a composite function (All boards)

$f(x) = x^2 - 4x + 9$, $x \in \mathbb{R}$, and $g(x) = \frac{2}{x + 2}$, $x > -2$.

(a) Find the range of f.
(b) Show that gf is defined and find its range.

$f(x) = (x - 2)^2 + 5 \geq 5$
So $f(x) \geq 5 > -2$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{2}{f(x) + 2}$, with $f(x) \geq 5$, so $f(x) + 2 \geq 7$

Largest value is $\frac{2}{7} = \frac{2}{7}$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 5$ (b) $0 < gf(x) \leq \frac{2}{7}$

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10 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially h = 25, and after 4 hours h = 1.

Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\frac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\, dh = \int -k\, dt$, so $2\sqrt{h} = -kt + C$

t = 0: $C = 2 \times 5 = 10$, so $2\sqrt{h} = 10 - kt$

t = 4, h = 1: $2 = 10 - 4k$, so k = 2

Empty when h = 0: $kt = 10$, so $t = \frac{10}{2} = 5$ hours

Answer: 5 hours