Day 173 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Separable differential equations (All boards)

Solve the differential equation

$y\frac{dy}{dx} = 2x$,

given that y = 2 when x = 0 (y > 0). Give y in terms of x.

Separate: $\int y\,dy = \int 2x\,dx$, so $\frac{y^2}{2} = 1x^2 + C$

x = 0, y = 2 gives C = 2

$y^2 = 2x^2 + 4$

Answer: $y = \sqrt{2x^2 + 4}$

2 Arithmetic sequences: finding a and d (All boards)

The 5th term of an arithmetic sequence is 15 and the 9th term is 31.
Find the first term and common difference, and the sum of the first 15 terms.

$u_9 - u_5 = 4\,d = 31 - 15$ so d = 4

$a + 4\,d = 15$ so $a = -1$

$S_{15} = \frac{15}{2}(2 \times (-1) + 14 \times 4)$

Answer: $a = -1$, $d = 4$, $S_{15} = 405$

3 Vertical motion under gravity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards at 14.7 m s^{-1} from a point h m above horizontal ground, and hits the ground 6 s later.
Find (a) h, (b) the greatest height above the ground. Take g = 9.8.

(a) Take upwards as positive: $s = -h$, $u = 14.7$, $a = -9.8$, $t = 6$

$-h = 14.7\times6 - 4.9\times6^2$, so h = 88.2

(b) At the highest point v = 0: $0 = 14.7^2 - 2\times9.8\times s$

s = 11.025 m above the point of projection

Greatest height above the ground = 88.2 + 11.025

Answer: (a) 88.2 m (b) 99.225 m

4 Correcting a recorded value (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

15 values have $\Sigma x = 570$ and $\Sigma x^2 = 21795$.
One value was recorded as 45 but should have been 35.
Find the correct mean and standard deviation (3 s.f.).

$\Sigma x = 570 - 45 + 35 = 560$

$\Sigma x^2 = 21795 - 45^2 + 35^2 = 20995$

Mean $= \frac{560}{15} = 37.3$

sd $= \sqrt{\frac{20995}{15} - (37.3)^2} = 2.43$

Answer: mean 37.3, sd 2.43

5 Optimisation problems (All boards)

A window is a rectangle of width 2x m and height h m, topped by a semicircle of radius x m. Its perimeter is 30 m.

Show that its area is $A = 30x - (2 + \frac{\pi}{2})x^2$, and find the maximum area to 3 s.f.

Perimeter: $2x + 2h + \pi x = 30$, so $h = \frac{30 - 2x - \pi x}{2}$

$A = 2xh + \frac{1}{2}\pi x^2 = x(30 - 2x - \pi x) + \frac{1}{2}\pi x^2 = 30x - (2 + \frac{\pi}{2})x^2$

$\frac{dA}{dx} = 30 - (4 + \pi)x = 0$, so $x = \frac{30}{4 + \pi} = 4.20$

$\frac{d^2A}{dx^2} = -(4 + \pi) < 0$, so a maximum; A = 63.0

Answer: A_max = 63.0 m^2 (x = 4.20)

6 Surds: equation with surd coefficients (AQA, Edexcel, OCR, MEI, Eduqas)

Solve $x(\sqrt{2} - 3) = -4 - \sqrt{2}$, writing x in the form $p + q\sqrt{2}$.

$x = \dfrac{-4 - \sqrt{2}}{\sqrt{2} - 3}$

Multiply top and bottom by $\sqrt{2} + 3$.

Numerator: $(-4 - \sqrt{2})(\sqrt{2} + 3) = -14 - 7\sqrt{2}$

Denominator: $2 - 9 = -7$

Answer: $x = 2 + \sqrt{2}$

7 Car and trailer (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 900 kg car tows a 300 kg trailer by a light tow-bar on a horizontal road. Driving force 980 N; resistances 400 N (car), 100 N (trailer). Find (a) the acceleration and the tow-bar tension. At 20 m s^{-1} the tow-bar breaks. (b) How far does the trailer then travel?

Whole system: $980 - 400 - 100 = 1200\,a$, so $a = 0.4$

Trailer: $T - 100 = 300 \times 0.4$, so $T = 100 + 300 \times 0.4 = 220$

(b) After the break the trailer decelerates at $100 \div 300 = \dfrac{1}{3}$ m s^{-2}

$0 = 20^2 - 2 \times \dfrac{1}{3} \times s$, so $s = 600$

Answer: (a) 0.4 m s^{-2}, 220 N (b) 600 m

8 Two-tailed binomial test critical region (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(28, p)$. H_0: $p = 0.2$ is tested against H_1: $p \neq 0.2$ at the 5% level.

(a) Find the critical region.

(b) Find the actual significance level of the test (3 s.f.).

(a) Each tail ≤ 0.025. $P(X \leq 1) = 0.0155$, $P(X \leq 2) = 0.0612$

$P(X \geq 11) = 0.0149$, $P(X \geq 10) = 0.0391$

Critical region: $X \leq 1$ or $X \geq 11$

(b) Actual level $= 0.0155 + 0.0149 = 0.0303$

Answer: (a) $X \leq 1$ or $X \geq 11$ (b) 0.0303

9 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially $h = 25$, and after 5 hours $h = 4$.

Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\dfrac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\,dh = \int -k\,dt$, so $2\sqrt{h} = -kt + C$

$t = 0$: $C = 2 \times 5 = 10$, so $2\sqrt{h} = 10 - kt$

$t = 5$, $h = 4$: $4 = 10 - 5k$, so $k = \dfrac{6}{5}$

Empty when $h = 0$: $kt = 10$, so $t = \dfrac{10}{\frac{6}{5}} = \dfrac{25}{3}$ hours

Answer: $\dfrac{25}{3}$ hours

10 Circle touching an axis (diagram) (AQA, Edexcel, OCR, MEI, Eduqas)

The diagram shows a circle that touches the x-axis at A(1, 0) and passes through B(−1, 2).

Find the equation of the circle.

Touches the x-axis at A, so the centre is $(1, r)$ and the radius is r.

B on the circle: $(-2)^2 + (2 - r)^2 = r^2$

$4 + 4 - 4r = 0$ so $r = 2$

Answer: $(x - 1)^2 + (y - 2)^2 = 4$