Day 169 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Normal distribution probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the journey time (minutes), is normally distributed with mean 39 and standard deviation 6.
Find (a) P(X > 30) (b) P(30 < X < 37.8), each to 3 s.f.

(a) $z = \frac{30 - 39}{6} = -1.50$, $P(Z > -1.50) = 0.933$

(b) $z_1 = -1.50$, $z_2 = \frac{37.8 - 39}{6} = -0.200$

$P(30 < X < 37.8) = \Phi(-0.200) - \Phi(-1.50) = 0.354$

Answer: (a) 0.933 (b) 0.354

2 Chain rule and gradients (All boards)

$y = \cos^2(x)$. Find $\frac{dy}{dx}$ and its exact value at $x = \frac{\pi}{3}$.

Let $u = \cos(x)$: $y = u^2$, so $\frac{dy}{dx} = 2u^1 \times \frac{du}{dx}$

$\frac{du}{dx} = -\sin(x)$, giving $\frac{dy}{dx} = -2\cos(x)\sin(x)$

At $x = \frac{\pi}{3}$: $\cos(x) = \frac{1}{2}$ and $\sin(x) = \frac{\sqrt{3}}{2}$, so gradient $= -\frac{\sqrt{3}}{2}$

Answer: $\frac{dy}{dx} = -2\cos(x)\sin(x)$; gradient $= -\frac{\sqrt{3}}{2}$

3 Forces in equilibrium with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

Forces $F_1 = (9\mathbf{i} + 5\mathbf{j})$ N, $F_2 = (5\mathbf{i} - 12\mathbf{j})$ N and $F_3 = (a\mathbf{i} + b\mathbf{j})$ N keep a 3 kg particle in equilibrium. Find (a) a and b. F_2 is then removed; (b) find the magnitude of the acceleration. Give answers to 3 s.f. if not exact.

(a) Equilibrium: $F_1 + F_2 + F_3 = 0$

i: $9 + 5 + a = 0$, so $a = -14$

j: $5 + (-12) + b = 0$, so $b = 7$

(b) The resultant is now $F_1 + F_3 = -F_2$, of magnitude $\sqrt{(5^2 + -12^2)} = 13$ N

$a = 13 \div 3$

Answer: (a) $a = -14$, $b = 7$ (b) 4.33 m s^{-2}

4 Quotient rule: stationary points (All boards)

Find the coordinates of the stationary points of

$y = \frac{x^2 + 24}{x + 1}$.

$\frac{dy}{dx} = \frac{(x + 1) \times 2x - (x^2 + 24) \times 1}{(x + 1)^2} = \frac{x^2 + 2x - 24}{(x + 1)^2}$

$\frac{dy}{dx} = 0$: $x^2 + 2x - 24 = 0$, so $(x - 4)(x + 6) = 0$

$x = 4$: $y = 8$; $x = -6$: $y = -12$

Answer: $(-6, -12)$, $(4, 8)$

5 Inverse of a rational function (All boards)

$f(x) = \frac{-x + 5}{x - 4}$, $x \in \mathbb{R}$, $x \neq 4$.

Find $f^{-1}(x)$ and state its domain.

$y = \frac{-x + 5}{x - 4}$: $y(x - 4) = -x + 5$

$xy - 4y = -x + 5$

$x(y + 1) = 4y + 5$

$x = \frac{4y + 5}{y + 1}$, then swap x and y

Answer: $f^{-1}(x) = \frac{4x + 5}{x + 1}$, $x \neq -1$

Day 169 **A Level Maths: A Tier 10-a-day: Model Solutions**

6 Bayes' theorem in context (All boards)

2% of people have a fault. The inspection test is positive for 90% of those who have it and for 5% of those who do not.
(a) Find the probability that a person tests positive.
(b) Given a positive test, find the probability that the person has the fault (3 s.f.).

(a) $P(+) = 0.02 \times 0.9 + 0.98 \times 0.05$
$= 0.018 + 0.049 = 0.067$

(b) $P(\text{has it} \mid +) = \dfrac{0.02 \times 0.9}{0.067}$

$= 0.269$

Answer: (a) 0.067 (b) 0.269

7 Integration by substitution (All boards)

Use the substitution $u = x + 4$ to find the exact value of

$\int_0^{32} \dfrac{x}{\sqrt{x+4}}\,dx$.

$x = u - 4$, $du = dx$; limits $u = 4$ to $u = 36$

$I = \int_4^{36} \dfrac{u-4}{\sqrt{u}}\,du = \int_4^{36} (u^{\frac{1}{2}} - 4u^{-\frac{1}{2}})\,du$

$= [\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}]$

$= \dfrac{320}{3}$

Answer: $\dfrac{320}{3}$

8 Recurrence relations: limits (All boards)

$u_1 = 12$ and $u_{n+1} = \frac{1}{3}u_n + 4$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.01$.

(a) $L = \frac{1}{3}L + 4$ gives $L = \dfrac{4}{1 - \frac{1}{3}} = 6$

(b) $u_n - L = (u_1 - L)\,r^{n-1} = 6 \times \frac{1}{3}^{n-1}$

$|6| \times \frac{1}{3}^{n-1} < \dfrac{1}{100}$

Check: n = 6 gives 0.02469, n = 7 gives 0.00823

Answer: (a) L = 6 (b) n = 7

9 Binomial expansion: a coefficient (All boards)

Find the coefficient of x^4 in the expansion of $(4 + x)^9$.

Term in x^4: ${}^9C_4 \times 4^5 \times (1x)^4$

$= 126 \times 1024 \times 1$

Answer: 129024

10 Closest approach of two particles (Edexcel, OCR, MEI)

At t = 0, P and Q have position vectors $(-23\mathbf{i} + 6\mathbf{j})$ m and $(-3\mathbf{i} - 4\mathbf{j})$ m and velocities $(6\mathbf{i} - 7\mathbf{j})$ m s^{-1} and $(3\mathbf{i} - 3\mathbf{j})$ m s^{-1}. Find (a) the time when they are closest, (b) the least distance between them.

$PQ = r_P - r_Q = (-20\mathbf{i} + 10\mathbf{j}) + t(3\mathbf{i} - 4\mathbf{j})$

$d^2 = (3t - 20)^2 + (-4t + 10)^2 = 25t^2 - 200t + 500$

(a) $d(d^2)/dt = 0$: $50\,t + -200 = 0$, so $t = 4$

(b) At $t = 4$: $PQ = -8\mathbf{i} - 6\mathbf{j}$, so $d = \sqrt{100}$

Answer: (a) t = 4 s (b) 10 m