Day 167 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Normal distribution probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the length (mm), is normally distributed with mean 80 and standard deviation 2.
Find (a) P(X > 83.6) (b) P(83.6 < X < 86.4), each to 3 s.f.

(a) $z = \frac{83.6 - 80}{2} = 1.80$, $P(Z > 1.80) = 0.0359$

(b) $z_1 = 1.80$, $z_2 = \frac{86.4 - 80}{2} = 3.20$

$P(83.6 < X < 86.4) = \Phi(3.20) - \Phi(1.80) = 0.0352$

Answer: (a) 0.0359 (b) 0.0352

2 Recurrence relations: limits (All boards)

$u_1 = 22$ and $u_{n+1} = \frac{1}{3} u_n + 12$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.001$.

(a) $L = \frac{1}{3} L + 12$ gives $L = \frac{12}{1 - \frac{1}{3}} = 18$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = 4 \times \left(\frac{1}{3}\right)^{n-1}$

$|4| \times \left(\frac{1}{3}\right)^{n-1} < \frac{1}{1000}$

Check: n = 8 gives 0.00183, n = 9 gives 0.00061

Answer: (a) L = 18 (b) n = 9

3 Trapezium rule with over or under estimate (AQA, Edexcel, OCR, MEI, Eduqas)

Use the trapezium rule with 4 strips to estimate $\int_1^3 5\left(\frac{1}{x}\right) dx$, giving your answer to 3 s.f.
State, with a reason, whether your estimate is an overestimate or an underestimate.

$h = \frac{1}{2}$; y-values at $x = 1, \frac{3}{2}, 2, \frac{5}{2}, 3$: 5.0000, 3.3333, 2.5000, 2.0000, 1.6667

$T = \frac{1}{4}[5.0000 + 1.6667 + 2(3.3333 + 2.5000 + 2.0000)]$

T = 5.58

The curve is convex (y'' > 0) on this interval, so the chords lie above the curve: overestimate

Answer: 5.58, an overestimate

4 Newton's second law with a time-varying force (AQA, Edexcel, OCR, MEI)

A 3 kg particle moves on a smooth horizontal line. At time t s a single force of (15 − 3t) N acts along the line. When t = 0 its speed is 2 m s^{-1}. Find (a) its speed when t = 1, (b) its greatest speed.

F = ma: $a = (-3t + 15) \div 3$

$v = \int a\, dt = 5t - 0.5t^2 + c$; v = 2 when t = 0

(a) v(1) = 6.5

(b) Greatest speed when a = 0: t = 5

v(5) = 14.5

Answer: (a) 6.5 m s^{-1} (b) 14.5 m s^{-1}

5 Binomial in two stages (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Bulbs are defective with probability 0.2, independently. A pack of 6 bulbs is rejected if it has at least 2 defective.
From 6 such packs, find the probability that exactly 4 are rejected (3 s.f.).

$Y \sim B(6, 0.2)$. $P(\text{rejected}) = P(Y \geq 2) = 1 - 0.655 = 0.345$

$W \sim B(6, 0.345)$ is the number rejected

$P(W = 4) = C(6, 4)(0.345)^4(1 - 0.345)^2 = 0.0909$

Answer: 0.0909

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6 Modulus inequalities (All boards)

Solve $|x + 4| < |-2x + 3|$.

Square both sides: $(x + 4)^2 < (-2x + 3)^2$
$((x + 4) - (-2x + 3))((x + 4) + (-2x + 3)) < 0$
$(3x + 1)(-x + 7) < 0$

Critical values $x = -\frac{1}{3}$ and $x = 7$; sketch the (positive or negative) quadratic.

Answer: $x < -\frac{1}{3}$ or $x > 7$

7 Position vector and velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P has position vector $\mathbf{r} = (t^3 - t)\mathbf{i} + (3t^2 - 6t)\mathbf{j}$ metres at time t s. Find (a) the speed of P when t = 2, (b) the value of t when P is moving parallel to $\mathbf{i}$. Give answers to 3 s.f. if not exact.

$\mathbf{v} = d\mathbf{r}/dt = (3t^2 - 1)\mathbf{i} + (6t - 6)\mathbf{j}$
(a) $t = 2$: $\mathbf{v} = 11\mathbf{i} + 6\mathbf{j}$
Speed $= \sqrt{(11^2 + 6^2)} = \sqrt{157}$
(b) Parallel to $\mathbf{i}$ means the $\mathbf{j}$ component is zero: $6t - 6 = 0$, so $t = 1$
Answer: (a) 12.5 m s^{-1} (b) $t = 1$

8 Optimisation problems (All boards)

An open-topped box has a square base of side x cm and volume 256 cm^3.

Show that its surface area is $A = x^2 + \frac{1024}{x}$, and find the least possible value of A, justifying that it is a minimum.

$h = \frac{256}{x^2}$, so $A = x^2 + 4xh = x^2 + \frac{1024}{x}$

$\frac{dA}{dx} = 2x - \frac{1024}{x^2} = 0$, so $x^3 = 512$ and $x = 8$

$\frac{d^2A}{dx^2} = 2 + \frac{2048}{x^3} > 0$, so a minimum

$A = 64 + \frac{1024}{8} = 192$

Answer: $A_{min} = 192$ cm^2 ($x = 8$)

9 Connected rates of change: cones and bowls (AQA, Edexcel, OCR, MEI, Eduqas)

Water pours into a cone, vertex down, at 50 cm^{3}s^{-1}. The surface radius is always $\frac{h}{2}$.

Find the rate at which the depth is rising when h = 5 cm.

$V = \frac{1}{3}\pi(h/2)^2 h = \frac{\pi h^3}{12}$, so $\frac{dV}{dh} = \frac{\pi h^2}{4}$

$\frac{dh}{dt} = \frac{dV}{dt \div dV/dh} = \frac{50 \times 4}{\pi h^2}$

$h = 5$: $\frac{dh}{dt} = \frac{8}{\pi}$ cm s^{-1}

Answer: $\frac{8}{\pi}$ cm/s

10 Circle: tangent at a point (AQA, Edexcel, OCR, MEI, Eduqas)

The circle $x^2 + y^2 - 4x + 6y - 12 = 0$ passes through P$(-1, -7)$.
(a) Find the equation of the tangent at P, in the form $ax + by = c$.
(b) Find the coordinates of the point where this tangent crosses the x-axis.

Centre $(2, -3)$. Gradient of radius CP = $\frac{4}{3}$

Gradient of tangent = $-\frac{3}{4}$

$y + 7 = -\frac{3}{4}(x + 1)$: $3x + 4y = -31$

$y = 0$: $x = -\frac{31}{3}$

Answer: (a) $3x + 4y = -31$ (b) $(-\frac{31}{3}, 0)$