Day 166 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Implicit differentiation: tangent to a curve (All boards)

A curve has equation $2x^2 - 2xy - 3y^2 + 3x + 3y = -4$.
Show that P(−2, 3) lies on the curve and find the equation of the tangent at P in the form ax + by + c = 0 with integer a, b, c.

Differentiate (product rule on the xy term): $(4x - 2y + 3) + (-2x - 6y + 3)\frac{dy}{dx} = 0$

At P(−2, 3): $-11 + (-11)\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -1$

Tangent: $y - 3 = (-1)(x + 2)$, i.e. $x + y - 1 = 0$

Answer: $x + y - 1 = 0$

2 Indices: solving equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4^{x-3} = 8^{x+1}$.

Write both sides as powers of 2.

$(2^2)^{x-3} = (2^3)^{x+1}$

$2(x - 3) = 3(x + 1)$

$2x - 6 = 3x + 3$

$-x = 9$

Answer: $x = -9$

3 Dividing a line in a ratio (2D position vectors) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B have position vectors $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$. The point P lies on AB with AP : PB = 2 : 1.

Find the position vector of P and the exact distance OP.

$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -9 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$OP = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$

Answer: $\begin{pmatrix} \mathbf{0} \\ \mathbf{-1} \end{pmatrix}$, OP = 1

4 Selections with combinations (OCR, MEI, Eduqas, CIE)

A group of 9 men and 3 women. A team of 4 is chosen at random.
Find the probability that the team contains at least one of the women.

Total teams = C(12, 4) = 495

Teams of all men = C(9, 4) = 126

P(at least one wome) = $1 - \frac{126}{495}$

$= \frac{41}{55}$

Answer: $\frac{41}{55}$

5 Connected rates of change: cones and bowls (AQA, Edexcel, OCR, MEI, Eduqas)

A hemispherical bowl of radius 5 cm is filled at 40 cm^3s^{-1}. When the depth is h, $V = \pi(5h^2 - \frac{1}{3}h^3)$.

Find $\frac{dh}{dt}$ when h = 4 cm.

$\frac{dV}{dh} = \pi(10h - h^2) = \pi h(10 - h)$

$\frac{dh}{dt} = \frac{40}{\pi h(10 - h)}$

h = 4: $\frac{dh}{dt} = \frac{40}{\pi \times 4 \times 6} = \frac{5}{3\pi}$ cm s^{-1}

Answer: $\frac{5}{3\pi}$ cm/s

6 Position vector and velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P has position vector $\mathbf{r} = (2t^3 - t)\mathbf{i} + (3t^2 - 6t)\mathbf{j}$ metres at time t s. Find (a) the speed of P when t = 3, (b) the value of t when P is moving parallel to $\mathbf{i}$. Give answers to 3 s.f. if not exact.

$\mathbf{v} = d\mathbf{r}/dt = (6t^2 - 1)\mathbf{i} + (6t - 6)\mathbf{j}$

(a) t = 3: $\mathbf{v} = 53\mathbf{i} + 12\mathbf{j}$

Speed = $\sqrt{(53^2 + 12^2)} = \sqrt{2953}$

(b) Parallel to $\mathbf{i}$ means the $\mathbf{j}$ component is zero: 6t − 6 = 0, so t = 1

Answer: (a) 54.3 m s^{-1} (b) t = 1

7 Expectation and variance of linear combinations (Edexcel, OCR, MEI, Eduqas, CIE)

X has E(X) = 2 and $\text{Var}(X) = \frac{5}{2}$.

(a) Find $E(X^2)$.
(b) Y = 3X + 7. Find E(Y) and Var(Y).
(c) X_1 and X_2 are independent copies of X. Find $\text{Var}(X_1 - X_2)$.

(a) $E(X^2) = \text{Var}(X) + [E(X)]^2 = \frac{5}{2} + 4 = \frac{13}{2}$

(b) $E(Y) = 3 \times 2 + 7 = 13$, $\text{Var}(Y) = (3)^2 \times \frac{5}{2} = \frac{45}{2}$

(c) $\text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 2 \times \frac{5}{2} = 5$

Answer: (a) $\frac{13}{2}$ (b) E(Y) = 13, Var(Y) = $\frac{45}{2}$ (c) 5

8 Modulus inequalities (All boards)

Solve $|x + 3| > |-3x - 3|$.

Square both sides: $(x + 3)^2 > (-3x - 3)^2$

$((x + 3) - (-3x - 3))((x + 3) + (-3x - 3)) > 0$

$(4x + 6)(-2x) > 0$

Critical values $x = -\frac{3}{2}$ and x = 0; sketch the (positive or negative) quadratic.

Answer: $-\frac{3}{2} < x < 0$

9 Tangent conditions (All boards)

Find the set of values of k for which the line $y = -3x + k$ meets the curve $y = 2x^2 - 6x - 3$ at two distinct points.

$2x^2 - 6x - 3 = -3x + k$ gives $2x^2 - 3x + (-3 - k) = 0$

Discriminant: $(-3)^2 - 4 \times 2(-3 - k) = 33 + 8k$

Two distinct points: discriminant > 0

Answer: $k > -\frac{33}{8}$

10 Projectile: perpendicular directions (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the ground at 40 m s^{-1} at angle α above the horizontal, $\tan\alpha = \frac{12}{5}$. At time T s, while in flight, it is moving perpendicular to its initial direction. Find (a) T, (b) the height of the ball then. Take g = 9.8. Give answers to 3 s.f. if not exact.

Initial velocity $(\frac{200}{13}, \frac{480}{13})$. At time T the velocity is $(\frac{200}{13}, \frac{480}{13} - 9.8T)$.

Perpendicular: scalar product = 0: $(\frac{200}{13})^2 + \frac{480}{13}(\frac{480}{13} - 9.8T) = 0$

$40^2 - 9.8T \times \frac{480}{13} = 0$ (using $\sin^2\alpha + \cos^2\alpha = 1$)

(a) $T = 40 \div (9.8 \times \frac{12}{13}) = \frac{650}{147}$

(b) $h = \frac{480}{13} \times \frac{650}{147} - 4.9 \times \frac{650}{147}^2 = \frac{4250}{63}$

Answer: (a) T = 4.42 s (b) 67.5 m