Day 160 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Circle: centre, radius and position of a point (AQA, Edexcel, OCR, MEI, Eduqas)

A circle has equation $x^2 + y^2 + 2x + 6y = 0$.
(a) Find the centre and radius of the circle.
(b) Determine whether P(4, −1) lies inside or outside the circle.

Complete the square: $(x + 1)^2 + (y + 3)^2 = 10$
Centre (−1, −3), radius $\sqrt{10}$
Distance^2 from centre to P = $5^2 + 2^2 = 29$
Compare with $r^2 = 10$
Answer: (a) centre (−1, −3), radius $\sqrt{10}$ (b) P is outside the circle

2 Continuous random variable: finding k and E(X) (CIE)

The random variable X has probability density function f(x) = kx(4 − x) for 0 ≤ x ≤ 4, and 0 otherwise.
(a) Find k. (b) Find E(X). (c) Find P(X > 2).

(a) $\int_0^4 f(x)\,dx = 1$ gives $\frac{32}{3}k = 1$, so $k = \frac{3}{32}$
(b) $E(X) = \int x\,f(x)\,dx = 2$
(c) $P(X > 2) = \int_2^4 f(x)\,dx = \frac{1}{2}$
Answer: (a) $k = \frac{3}{32}$ (b) 2 (c) $\frac{1}{2}$

3 Equations with sin 2x and a trig function (All boards)

Solve $\sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$.

$\sin 2\theta \equiv 2\sin\theta\cos\theta$: $\sin\theta(2\cos\theta - 1) = 0$
$\sin\theta = 0$: $\theta = 0, \pi, 2\pi$
$\cos\theta = \frac{1}{2}$: $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
Do not divide by $\sin\theta$, as that would lose solutions
Answer: $\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

4 Proof by exhaustion (AQA, Edexcel, OCR, MEI, Eduqas)

Prove that the square of any integer is either a multiple of 3 or one more than a multiple of 3.

Any integer has the form 3k, 3k + 1 or 3k + 2.
$(3k)^2 = 9k^2 = 3(3k^2)$
$(3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$
$(3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$
All cases covered.
Answer: squares are of the form 3m or 3m + 1

5 Regression line from summary data (AQA, Edexcel, OCR, MEI, Eduqas)

n = 9 pairs of (x, y): $\Sigma x = 105$, $\Sigma y = 408$, $\Sigma x^2 = 1641$, $\Sigma xy = 5523$.
(a) Find the equation of the regression line of y on x (3 s.f.).
(b) Estimate y when x = 15.

$S_{xx} = 1641 - \frac{105^2}{9} = 416$, $S_{xy} = 5523 - \frac{105 \times 408}{9} = 763$

$b = \frac{S_{xy}}{S_{xx}} = 1.83$
$a = \bar{y} - b\,\bar{x} = 45.3 - (1.83)(11.7) = 23.9$
(b) $y = 23.9 + (1.83)(15) = 51.4$
Answer: (a) $y = 23.9 + 1.83x$ (b) 51.4

6 Binomial expansion: rational powers (All boards)

Expand $(1 - \frac{x}{2})^{\frac{3}{2}}$ in ascending powers of x up to and including the term in x^2.

State the range of values of x for which the expansion is valid.

$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \ldots$ with $n = \frac{3}{2}$, $y = -\frac{x}{2}$

Coefficients of 1, x, x^2: 1, $-\frac{3}{4}$, $\frac{3}{32}$

Valid when $|\frac{x}{2}| < 1$, i.e. $|x| < 2$

Answer: $1 - \frac{3}{4}x + \frac{3}{32}x^2$; $|x| < 2$

7 Velocity, rest and total distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on the x-axis with velocity $v = (3t^2 - 12t + 9)$ m s^{-1} at time t s, $t \geq 0$, and starts at O. Find (a) the times when P is at instantaneous rest, (b) the total distance P travels in the first 5 s.

(a) $v = 3(t - 1)(t - 3) = 0$, so $t = 1$ or $t = 3$

(b) Displacement $x = \int v\,dt = t^3 - 6t^2 + 9t$

$x(1) = 4$, $x(3) = 0$, $x(5) = 20$

Direction changes at t = 1 and t = 3, so distance = $|4| + |0 - 4| + |20 - 0|$

Answer: (a) t = 1, t = 3 (b) 28 m

8 Greatest and least values using R-form (All boards)

$g(\theta) = \frac{1}{20 + 15\sin\theta + 8\cos\theta}$

(a) Write $15\sin\theta + 8\cos\theta$ in the form $R\sin(\theta + \alpha)$, with α acute.

(b) Find the greatest and least values of $g(\theta)$ and the smallest positive θ at which the greatest value occurs (1 d.p.).

$R = \sqrt{15^2 + 8^2} = 17$, $\alpha = \tan^{-1}\frac{8}{15} = 28.1°$

Denominator $= 20 + 17\sin(\theta + 28.1°)$, ranging from 3 to 37

Greatest $g = \frac{1}{3}$ (denominator least), least $g = \frac{1}{37}$

Least denominator when $\sin(\theta + 28.1°) = -1$: $\theta + 28.1° = 270°$

$\theta = 241.9°$

Answer: R = 17, α = 28.1°; greatest $\frac{1}{3}$, least $\frac{1}{37}$; θ = 241.9°

9 Integration by parts (All boards)

Use integration by parts, with $u = \ln x$ and $\frac{dv}{dx} = 1$, to find the exact value of

$\int_1^4 \ln x\,dx$.

$\int \ln x\,dx = x\ln x - \int x \times \frac{1}{x}\,dx = x\ln x - x$

$[x\ln x - x]$ from 1 to 4

$= 8\ln 2 - 3$

Answer: $8\ln 2 - 3$

10 Ladder in limiting equilibrium (AQA, Edexcel, OCR, MEI)

A uniform ladder AB (13 m, 30 kg) rests with A on rough ground, $\mu = \frac{1}{3}$, and B against a smooth wall, B being 12 m above the ground. A man of mass 90 kg climbs the ladder. How far along the ladder from A can he climb before it slips?

Foot of ladder is $\sqrt{(13^2 - 12^2)} = 5$ m from the wall

Vertical: $R = (30 + 90)g$. Limiting friction: $F = \mu R = \frac{1}{3}(30 + 90)g$

Horizontal: $S = F$ (wall reaction S)

Moments about A (man at distance x along ladder, horizontal distance $x \times 5/13$):

$S \times 12 = 30\,g \times 2.5 + 90\,g \times (5x \div 13)$

$40 \times 12 = 75 + \frac{450}{13}x$ (g cancels), so $x = 11.7$

Answer: 11.7 m