Day 147 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Arc length and sector area (All boards)

A sector of a circle has area 16 cm^2 and angle 0.5 radians.
Find the radius, the arc length and the perimeter of the sector.

Area = $\frac{1}{2}r^2\theta$: $16 = \frac{1}{2}r^2 \times 0.5$, so $r^2 = 64 = 8^2$

r = 8 cm; arc length $s = r\theta = 8 \times 0.5 = 4$ cm
Perimeter = 2r + s = 16 + 4 = 20 cm
Answer: r = 8 cm, arc 4 cm, perimeter 20 cm

2 Range of horizontal force on a rough slope (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 2 kg particle is in equilibrium on a rough plane inclined at α, $\tan\alpha = \frac{15}{8}$, $\mu = \frac{1}{3}$. A horizontal force of P N, in the vertical plane through a line of greatest slope, pushes it towards the plane. Find the least and greatest values of P. Take g = 9.8. Give answers to 3 s.f. if not exact.

$\sin\alpha = \frac{15}{17}$, $\cos\alpha = \frac{8}{17}$, weight W = 19.6 N. Normal reaction $R = W\cos\alpha + P\sin\alpha$
Least P: about to slide down, friction $F = \mu R$ up the slope
$P\cos\alpha + \mu(W\cos\alpha + P\sin\alpha) = W\sin\alpha \rightarrow P = W(\sin\alpha - \mu\cos\alpha) \div (\cos\alpha + \mu\sin\alpha) = 18.6$
Greatest P: about to slide up, friction down the slope
$P\cos\alpha = W\sin\alpha + \mu(W\cos\alpha + P\sin\alpha) \rightarrow P = W(\sin\alpha + \mu\cos\alpha) \div (\cos\alpha - \mu\sin\alpha) = 115$
Answer: least 18.6 N, greatest 115 N

3 Completing the square (All boards)

$f(x) = -2x^2 - 8x - 3$.
(a) Write f(x) in the form $a(x + p)^2 + q$.
(b) State the maximum value of f(x) and the value of x at which it occurs.
(c) Solve f(x) = 0, giving surds in simplest form.

$f(x) = -2(x + 2)^2 + 5$
(b) maximum value 5 when x = −2
(c) $-2(x + 2)^2 = -5$, so $(x + 2)^2 = \frac{5}{2}$

$x + 2 = \pm\sqrt{\frac{5}{2}}$

Answer: (a) $-2(x + 2)^2 + 5$ (b) maximum 5 at x = −2 (c) $x = -2 \pm \frac{\sqrt{10}}{2}$

4 Inverse trigonometric functions (AQA, Edexcel, OCR, MEI, Eduqas)

Find the exact value of $\sin(2\arccos\frac{12}{13})$.

Let $\theta = \arccos\frac{12}{13}$, so $\cos\theta = \frac{12}{13}$ with θ acute

$\sin\theta = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$

$\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{5}{13} \times \frac{12}{13} = \frac{120}{169}$

Answer: $\frac{120}{169}$

5 Parametric equations: points on axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2t^2 + 4$, $y = -2t(t^2 - 1)$.
Find the coordinates of the points where the curve meets the x-axis.

y = 0: t = 0 or $t^2 = 1$, so t = 0, 1, −1
t = 0: x = 4
t = ±1: x = 6
Answer: (4, 0) and (6, 0)

6 Work-energy on a rough slope (CIE)

A 2 kg block is released from rest on a rough plane inclined at α, $\tan\alpha = \frac{4}{3}$, $\mu = 0.5$, and slides 10 m down a line of greatest slope. Use work and energy to find (a) the work done against friction, (b) its speed after 10 m. Take $g = 10$.

$R = 2g\cos\alpha = 12$ N, $F = \mu R = 6$ N
(a) Work done against friction $= F \times d = 6 \times 10$
(b) Loss of PE $= 2g \times 10\sin\alpha = 160$ J
Work-energy: $\frac{1}{2}\times 2 \times v^2 = 160 - 60 = 100$
$v^2 = 100$
Answer: (a) 60 J (b) 10 m s^{-1}

7 Conditional probability without replacement (All boards)

A bag has 6 red and 3 blue counters. Two are taken at random without replacement.
Given that the counters are different colours, find the probability that the first is red.

$P(\text{red then blue}) = \frac{6}{9} \times \frac{3}{8} = \frac{1}{4}$

$P(\text{blue then red}) = \frac{3}{9} \times \frac{6}{8} = \frac{1}{4}$

$P(\text{different}) = \frac{1}{2}$

$P(\text{first red} \mid \text{different}) = \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{2}$

Answer: $\frac{1}{2}$

8 Optimisation problems (All boards)

A window is a rectangle of width $2x$ m and height h m, topped by a semicircle of radius x m. Its perimeter is 60 m.
Show that its area is $A = 60x - (2 + \frac{\pi}{2})x^2$, and find the maximum area to 3 s.f.

Perimeter: $2x + 2h + \pi x = 60$, so $h = \frac{60 - 2x - \pi x}{2}$

$A = 2xh + \frac{1}{2}\pi x^2 = x(60 - 2x - \pi x) + \frac{1}{2}\pi x^2 = 60x - (2 + \frac{\pi}{2})x^2$

$\frac{dA}{dx} = 60 - (4 + \pi)x = 0$, so $x = \frac{60}{4 + \pi} = 8.40$

$\frac{d^2A}{dx^2} = -(4 + \pi) < 0$, so a maximum; $A = 252$

Answer: A_max = 252 m^2 (x = 8.40)

9 Conditional binomial probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(9, 0.2)$. Find $P(X = 4 \mid X \geq 3)$ to 3 significant figures.

$P(X = 4 \mid X \geq 3) = \frac{P(X = 4)}{P(X \geq 3)}$ (since $X = 4$ implies $X \geq 3$)
$P(X = 4) = 0.0661$
$P(X \geq 3) = 1 - P(X \leq 2) = 0.262$
Ratio $= 0.252$
Answer: 0.252

10 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas)

A circle touches both coordinate axes and passes through the point (6, −3). Its centre lies in the fourth quadrant. Find the equations of the two possible circles.

Centre (r, −r), radius r > 0.

Through (6, −3): $(6 - r)^2 + (-3 + r)^2 = r^2$

$r^2 - 18r + 45 = 0$

$(r - 3)(r - 15) = 0$, so r = 3 or 15

Answer: $(x - 3)^2 + (y + 3)^2 = 9$ or $(x - 15)^2 + (y + 15)^2 = 225$