Day 146 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Scalar product, magnitude of a sum (CIE (9709); Further Maths on other boards)

$|a| = 5$, $|b| = 2$ and the angle between a and b is 135°.
Find the exact values of $a \cdot b$ and $|a + b|$.

$a \cdot b = |a||b|\cos 135° = 5 \times 2 \times -\frac{\sqrt{2}}{2} = -5\sqrt{2}$

$|a + b|^2 = |a|^2 + 2a \cdot b + |b|^2 = 25 + -10\sqrt{2} + 4 = 29 - 10\sqrt{2}$

$|a + b| = \sqrt{29 - 10\sqrt{2}}$

Answer: $a \cdot b = -5\sqrt{2}$, $|a + b| = \sqrt{29 - 10\sqrt{2}}$

2 Circle: centre, radius and position of a point (AQA, Edexcel, OCR, MEI, Eduqas)

A circle has equation $x^2 + y^2 - 4x + 2y - 29 = 0$.
(a) Find the centre and radius of the circle.
(b) Determine whether P(1, 3) lies inside or outside the circle.

Complete the square: $(x - 2)^2 + (y + 1)^2 = 34$

Centre $(2, -1)$, radius $\sqrt{34}$

Distance^2 from centre to P $= (-1)^2 + 4^2 = 17$

Compare with $r^2 = 34$

Answer: (a) centre $(2, -1)$, radius $\sqrt{34}$ (b) P is inside the circle

3 Stopping distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A driver at constant speed sees a hazard 41.6 m ahead. After a reaction time of 1 s she brakes with deceleration 5 m s^{-2} and stops exactly at the hazard. Find (a) her initial speed, (b) the total time from seeing the hazard to stopping.

Let the initial speed be v. Reaction distance = 1 v, braking distance = $v^2 \div (2 \times 5)$

$1\,v + v^2 \div 10 = 41.6$, so $v^2 + 10\,v - 416 = 0$

$(v - 16)(v + 26) = 0$, so $v = 16$

(b) Braking time = $16 \div 5 = 3.2$ s; total = 1 + 3.2

Answer: (a) 16 m s^{-1} (b) 4.2 s

4 Outliers (AQA, Edexcel, OCR, MEI, Eduqas)

A data set has lower quartile 35 and upper quartile 44.
A value is an outlier if it is more than 1.5 × IQR above Q_3 or below Q_1.
Which of 19, 24, 36, 61 are outliers?

IQR = 44 − 35 = 9, so 1.5 × IQR = 13.5

Lower limit = 35 − 13.5 = 21.5

Upper limit = 44 + 13.5 = 57.5

Outliers are values below 21.5 or above 57.5

Answer: 19, 61

5 Partial fractions: improper fraction (All boards)

Express $\frac{x^2 - 2x - 6}{(x - 4)(x - 3)}$ in the form $1 + \frac{A}{(x - 4)} + \frac{B}{(x - 3)}$, where A and B are constants to be found.

Divide: $x^2 - 2x - 6 = (x^2 - 7x + 12) + (5x - 18)$

So the fraction is $1 + \frac{5x - 18}{(x - 4)(x - 3)}$

$A(x - 3) + B(x - 4) \equiv 5x - 18$

$x = 4$ gives $A = 2$; $x = 3$ gives $B = 3$

Answer: A = 2, B = 3

6 Parametric differentiation with trigonometry (All boards)

A curve has parametric equations $x = 4\sin t$, $y = \cos 2t$.

Find $\frac{dy}{dx}$ in terms of t and the exact equation of the tangent at the point where $t = \frac{\pi}{6}$, in the form $y = mx + c$.

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = -\frac{2\sin 2t}{4\cos t}$

$t = \frac{\pi}{6}$: gradient $m = -\frac{1}{2}$, point $(2, \frac{1}{2})$

$c = y - mx = \frac{3}{2}$

Answer: $\frac{dy}{dx} = -\frac{2\sin 2t}{4\cos t}$; $y = -\frac{1}{2}x + \frac{3}{2}$

7 Venn diagram with algebra (All boards)

The Venn diagram shows the numbers of 40 students in sets A and B.

(a) Find the value of x.

(b) A student in B is chosen at random. Find P(A).

(a) Total: $x + x + (3x + 2) + 3 = 40$

$5x + 5 = 40$, so $x = 7$

(b) Regions: 7, 7, 23, 3

In B: $7 + 23 = 30$, of which in A: 7

Answer: (a) x = 7 (b) $\frac{7}{30}$

8 Foot of perpendicular and distance to a line (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 12 \\ 6 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$.

Find the coordinates of the foot of the perpendicular from P(4, −4, 2) to l, and the exact distance from P to l.

Foot F = A + t d: $\mathbf{PF} = \begin{pmatrix} -3t + 8 \\ -3t + 10 \\ t - 3 \end{pmatrix}$

$\mathbf{PF} \cdot \mathbf{d} = 0$ gives $19t - 57 = 0$, so $t = 3$

F = (3, −3, 2); $\mathbf{PF} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, so $|\mathbf{PF}| = \sqrt{2}$

Answer: F(3, −3, 2), distance $\sqrt{2}$

9 Circle through two points with centre on a line (AQA, Edexcel, OCR, MEI, Eduqas)

A circle passes through A(3, 9) and B(−7, −15).

Its centre lies on the line $y = -x - 5$.

Find the equation of the circle.

Centre $(h, -h - 5)$ is equidistant from A and B.

$(h - 3)^2 + (-h - 14)^2 = (h + 7)^2 + (-h + 10)^2$

Solving gives $h = -2$, so centre $(-2, -3)$

$r^2 = (5)^2 + (12)^2 = 169$

Answer: $(x + 2)^2 + (y + 3)^2 = 169$

10 Ladder in limiting equilibrium (AQA, Edexcel, OCR, MEI)

A uniform ladder AB (5 m, 15 kg) rests with A on rough ground, $\mu = 0.25$, and B against a smooth wall, B being 4 m above the ground. A man of mass 75 kg climbs the ladder. How far along the ladder from A can he climb before it slips?

Foot of ladder is $\sqrt{(5^2 - 4^2)} = 3$ m from the wall

Vertical: $R = (15 + 75)g$. Limiting friction: $F = \mu R = 0.25(15 + 75)g$

Horizontal: $S = F$ (wall reaction S)

Moments about A (man at distance x along ladder, horizontal distance $x \times 3/5$):

$S \times 4 = 15\,g \times 1.5 + 75\,g \times (3x \div 5)$

$22.5 \times 4 = 22.5 + 45x$ (g cancels), so $x = 1.5$

Answer: 1.5 m