Day 14 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Poisson probabilities (CIE)

Emails arrive at a mean rate of 6 per hour, as a Poisson process.
Find the probability that in one such period there are
(a) exactly 8 (b) at least 8 (c) fewer than 8 (3 s.f.).

$X \sim Po(6)$

(a) $P(X = 8) = e^{-6}\, 6^8/40320! = 0.103$
(b) $1 - P(X \leq 7) = 1 - 0.744 = 0.256$
(c) $P(X \leq 7) = 0.744$
Answer: (a) 0.103 (b) 0.256 (c) 0.744

2 Stopping distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A driver at constant speed sees a hazard 60 m ahead. After a reaction time of 0.5 s she brakes with deceleration 4 m s^{-2} and stops exactly at the hazard. Find (a) her initial speed, (b) the total time from seeing the hazard to stopping.

Let the initial speed be v. Reaction distance = 0.5 v, braking distance = $v^2 \div (2\times4)$
$0.5\,v + v^2 \div 8 = 60$, so $v^2 + 4\,v - 480 = 0$
$(v - 20)(v + 24) = 0$, so $v = 20$
(b) Braking time = $20 \div 4 = 5$ s; total = 0.5 + 5
Answer: (a) 20 m s^{-1} (b) 5.5 s

3 Differentiating a^x (All boards)

$y = 3^{2x}$.
The gradient of the curve at P is 162 ln 3. Find the x-coordinate of P.

$\frac{dy}{dx} = 2 \ln 3 \times 3^{2x}$

$2 \ln 3 \times 3^{2x} = 162 \ln 3$, so $3^{2x} = 81$
$2x = 4$, so $x = 2$
Answer: x = 2

4 Perpendicular bisector and area (All boards)

A is (5, −3) and B is (−1, −1).
(a) Find the equation of the perpendicular bisector of AB, in the form ax + by = c.
(b) The perpendicular bisector meets the x-axis at P. Find the area of triangle ABP.

Midpoint of AB = (2, −2); gradient AB = $-\frac{1}{3}$

Perpendicular gradient = 3
Equation: $3x - y = 8$

y = 0: P = $(\frac{8}{3}, 0)$

Area = ½ $|(-1 - 5)(0 - (-3)) - (\frac{8}{3} - 5)(-1 - (-3))|$

Answer: (a) $3x - y = 8$ (b) $\frac{20}{3}$

5 Chain rule and gradients (All boards)

$y = \ln(x^2 + 6)$.
Find $\frac{dy}{dx}$ and the gradient of the curve at x = 3.

$\frac{dy}{dx} = \frac{1}{x^2 + 6} \times 2x = \frac{2x}{x^2 + 6}$

At x = 3: gradient = $\frac{6}{15} = \frac{2}{5}$

Answer: $\frac{dy}{dx} = \frac{2x}{x^2 + 6}$; gradient = $\frac{2}{5}$

6 Periodic sequences (All boards)

A sequence is defined by $u_1 = 3$ and $u_{n+1} = \frac{1}{1 - u_n}$.

(a) Find u_2 and u_3.

(b) Find u_{101} and $\sum_{r=1}^{101} u_r$.

$u_2 = -\frac{1}{2}$, $u_3 = \frac{2}{3}$, $u_4 = 3$

$u_4 = u_1$, so the sequence repeats with period 3.

$101 = 3 \times 33 + 2$, so $u_{101} = u_2 = -\frac{1}{2}$

Sum of one period = $\frac{19}{6}$; 33 full periods plus 2 extra term(s)

Answer: (a) $-\frac{1}{2}$, $\frac{2}{3}$ (b) $u_{101} = -\frac{1}{2}$, sum = 107

7 Critical values for a binomial distribution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(30, 0.25)$.

Find the largest value of c such that $P(X \leq c) < 0.01$.

$P(X \leq 1) = 0.00196$
$P(X \leq 2) = 0.0106$
$0.00196 < 0.01$ but $0.0106 > 0.01$
Answer: c = 1

8 Surds: equation with surd coefficients (AQA, Edexcel, OCR, MEI, Eduqas)

Solve $x(\sqrt{3} - 3) = -1 + \sqrt{3}$, writing x in the form $p + q\sqrt{3}$.

$x = \frac{-1 + \sqrt{3}}{\sqrt{3} - 3}$

Multiply top and bottom by $\sqrt{3} + 3$.
Numerator: $(-1 + \sqrt{3})(\sqrt{3} + 3) = 2\sqrt{3}$
Denominator: $3 - 9 = -6$

Answer: $x = -\frac{1}{3}\sqrt{3}$

9 Separable differential equations (All boards)

Solve the differential equation

$y\frac{dy}{dx} = e^x$,

given that $y = 4$ when $x = 0$ ($y > 0$). Give y in terms of x.

Separate: $\int y\,dy = \int e^x\,dx$, so $\frac{y^2}{2} = e^x + C$

$x = 0$, $y = 4$ gives $C = 7$
$y^2 = 2e^x + 14$

Answer: $y = \sqrt{2e^x + 14}$

10 Projectile through a given point (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at angle α above the horizontal, $\tan\alpha = \frac{3}{4}$, and passes through A, 16 m horizontally from O and 9.5 m above the ground. Find the speed of projection. Take $g = 9.8$.

Let the speed be U. Horizontal: $16 = U\cos\alpha \times t$, so $t = 16 \div (0.8\,U)$
Vertical: $9.5 = U\sin\alpha \times t - 4.9t^2$
Eliminate t: $y = x\tan\alpha - 4.9x^2 \div (U^2\cos^2\alpha)$
$9.5 = 12 - 4.9 \times 16^2 \div (0.64\,U^2)$
$U^2 = 1960 \div 2.5 = 784$, $U = 28$

Answer: 28 m s^{-1}