Day 138 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Circle with a given diameter (AQA, Edexcel, OCR, MEI, Eduqas)

A(10, −10) and B(−2, 6) are the ends of a diameter of a circle.
(a) Find the equation of the circle, in the form $x^2 + y^2 + ax + by + c = 0$.
(b) Show that C(4, 8) lies on the circle.

Centre = midpoint of AB = (4, −2)
$r^2 = (6)^2 + (-8)^2 = 100$
$(x - 4)^2 + (y + 2)^2 = 100$: $x^2 + y^2 - 8x + 4y - 80 = 0$
(b) $CM^2 = (0)^2 + (10)^2 = 100 = r^2$
Answer: (a) $x^2 + y^2 - 8x + 4y - 80 = 0$

2 Exact definite integrals of standard functions (All boards)

Find the exact value of

$\int_0^{\frac{\pi}{2}} (4\sin 2x + 6\cos x)\,dx$.

Integrate: $[-2\cos 2x + 6\sin x]$
Substitute the limits and subtract: 10
Answer: 10

3 Forces in equilibrium with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

Forces $F_1 = (-6\mathbf{i} - 2\mathbf{j})$ N, $F_2 = (15\mathbf{i} - 8\mathbf{j})$ N and $F_3 = (a\mathbf{i} + b\mathbf{j})$ N keep a 5 kg particle in equilibrium. Find (a) a and b. F_2 is then removed; (b) find the magnitude of the acceleration.

(a) Equilibrium: $F_1 + F_2 + F_3 = 0$
i: −6 + 15 + a = 0, so a = −9
j: −2 + (−8) + b = 0, so b = 10
(b) The resultant is now $F_1 + F_3 = -F_2$, of magnitude $\sqrt{(15^2 + -8^2)} = 17$ N
a = 17 ÷ 5
Answer: (a) a = −9, b = 10 (b) 3.4 m s^{-2}

4 Finding an unknown standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X is normally distributed with mean 68 and standard deviation σ.
Given that P(X < 64) = 0.3085, find σ to 3 significant figures.

P(Z < z) = 0.3085, so z = −0.500

$$\frac{64 - 68}{\sigma} = -0.500$$

$$\sigma = \frac{4}{0.500} = 8.00$$

Answer: σ = 8.00

5 Non-uniform rod and equal tensions (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB (length 4 m, mass 4 kg) has its centre of mass at G, AG = 1.25 m. It hangs horizontally from vertical strings at A and B. A particle of mass m kg is fixed at P, AP = 3 m, making the tensions equal. Find (a) m, (b) the tension. Take g = 9.8.

Equal tensions T: 2T = (4 + m)g, so T = ½(4 + m)g
Moments about A: T×4 = 4 g×1.25 + m g×3
½(4 + m)×4 = 5 + 3 m → 8 + 2 m = 5 + 3 m
m (3 − 2) = 8 − 5, so m = 3
(b) T = ½(4 + 3)×9.8
Answer: (a) m = 3 (b) 34.3 N

6 Change of sign to locate a root (All boards)

$f(x) = x^2 - 3\cos x - 7$.
Show that the equation $f(x) = 0$ has a root α in the interval $[2, 3]$ (x in radians).

$f(x)$ is continuous on the interval
$f(2) = -1.75 < 0$
$f(3) = 4.97 > 0$
A change of sign, so a root lies in $[2, 3]$
Answer: Sign change: $f(2) = -1.75$, $f(3) = 4.97$

7 Bayes' theorem in context (All boards)

10% of people have a fault. The inspection test is positive for 80% of those who have it and for 5% of those who do not.
(a) Find the probability that a person tests positive.
(b) Given a positive test, find the probability that the person has the fault (3 s.f.).

(a) $P(+) = 0.1 \times 0.8 + 0.9 \times 0.05$
$= 0.08 + 0.045 = 0.125$

(b) $P(\text{has it} \mid +) = \dfrac{0.1 \times 0.8}{0.125}$

$= 0.640$
Answer: (a) 0.125 (b) 0.640

8 Proof by contradiction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove by contradiction that there are no integers m and n such that $6m + 15n = 98$.

Assume $6m + 15n = 98$ for integers m, n.
Then $3(2m + 5n) = 98$, so 3 divides 98.
But 98 is not a multiple of 3. Contradiction.
Answer: no such integers exist

9 Inverse of a quadratic function (All boards)

$f(x) = x^2 - 4x + 9$, $x \geq 2$.
Find $f^{-1}(x)$ and state its domain.

$f(x) = (x - 2)^2 + 5$
Let $y = f(x)$: $(x - 2)^2 = y - 5$
$x \geq 2$, so take the positive root: $x = 2 + \sqrt{y - 5}$
Answer: $f^{-1}(x) = 2 + \sqrt{x - 5}$, $x \geq 5$

10 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially $h = 36$, and after 6 hours $h = 16$.
Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\dfrac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\, dh = \int -k\, dt$, so $2\sqrt{h} = -kt + C$

$t = 0$: $C = 2 \times 6 = 12$, so $2\sqrt{h} = 12 - kt$

$t = 6$, $h = 16$: $8 = 12 - 6k$, so $k = \dfrac{2}{3}$

Empty when $h = 0$: $kt = 12$, so $t = \dfrac{12}{\frac{2}{3}} = 18$ hours

Answer: 18 hours