Day 135 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Horizontal projection (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected horizontally at 6 m s^{-1} from a point 44.1 m above horizontal ground. Find (a) the horizontal distance to the landing point, (b) the speed and direction (angle below the horizontal, 1 d.p.) on landing. Take g = 9.8. Give answers to 3 s.f. if not exact.

Vertical: $44.1 = \frac{1}{2} \times 9.8 \times t^2$, so t = 3

(a) Horizontal distance = 6 × 3

(b) Vertical speed on landing = 9.8×3 = 29.4; horizontal speed = 6

Speed = $\sqrt{(6^2 + 29.4^2)} = 30.0$

$\tan\theta = 29.4 \div 6$, $\theta = 78.5°$

Answer: (a) 18 m (b) 30.0 m s^{-1} at 78.5° below the horizontal

2 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has P(X = x) = k(7 − x) for x = 2, 3, 4, 5.

(a) Show that $k = \frac{1}{14}$. (b) Find E(X) and Var(X).

(c) Find P(X ≥ 4).

(a) Σ P(X = x) = 1: 5 + 4 + 3 + 2 = 14, so 14k = 1

(b) $E(X) = 2 \times \frac{5}{14} + 3 \times \frac{2}{7} + 4 \times \frac{3}{14} + 5 \times \frac{1}{7} = \frac{22}{7}$

$E(X^2) = 11$, so $Var(X) = 11 - (\frac{22}{7})^2 = \frac{55}{49}$

(c) $P(X \geq 4) = \frac{3}{14} + \frac{1}{7} = \frac{5}{14}$

Answer: (b) $E(X) = \frac{22}{7}$, $Var(X) = \frac{55}{49}$ (c) $\frac{5}{14}$

3 Constant acceleration (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cyclist passes P with speed 7 m s^{-1} and accelerates uniformly. She travels 100 m in the next 10 s, reaching Q. Find (a) her acceleration, (b) her speed at Q, (c) her speed at the midpoint of PQ. Give answers to 3 s.f. if not exact.

(a) $s = ut + \frac{1}{2}at^2$: $100 = 7 \times 10 + \frac{1}{2} \times a \times 10^2$, so a = 0.6

(b) v = u + at = 7 + 0.6×10 = 13

(c) For the midpoint, s = 50: $v^2 = u^2 + 2as = 7^2 + 2 \times 0.6 \times 50$

$v^2 = 109$

Answer: (a) 0.6 m s^{-2} (b) 13 m s^{-1} (c) 10.4 m s^{-1}

4 Fixed point iteration (All boards)

The equation $e^x = 6x + 5$ has a root α. Use the iteration $x_{n+1} = \ln(6x + 5)$ with $x_0 = 2$ to find x_1, x_2 and x_3 to 4 d.p., and hence the value of α to 2 d.p. (continue until the 2 d.p. value is settled).

$x_1 = \ln(6 \times 2 + 5) = 2.8332$

$x_2 = 3.0910$, $x_3 = 3.1590$

Continuing, the values settle at 3.1818, so α = 3.18

Answer: x_1 = 2.8332, x_2 = 3.0910, x_3 = 3.1590; α = 3.18

5 Inverse of a rational function (All boards)

$f(x) = \frac{-2x + 4}{4x + 4}$, $x \in \mathbb{R}$, $x \neq -1$.

Find $f^{-1}(x)$ and state its domain.

$y = \frac{-2x + 4}{4x + 4}$: $y(4x + 4) = -2x + 4$

$4xy + 4y = -2x + 4$

$x(4y + 2) = -4y + 4$

$x = \frac{-4y + 4}{4y + 2}$, then swap x and y

Answer: $f^{-1}(x) = \frac{-4x + 4}{4x + 2}$, $x \neq -\frac{1}{2}$

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6 Binomial mean and variance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X has a binomial distribution with mean 25 and variance 12.5.
(a) Find n and p.
(b) Find P(X = 23) to 3 significant figures.

(a) np = 25 and npq = 12.5

$q = \frac{npq}{np} = \frac{12.5}{25} = 0.5$, so p = 0.5

$n = \frac{25}{0.5} = 50$

(b) $P(X = 23) = C(50, 23)(0.5)^{23}(0.5)^{27} = 0.0960$

Answer: (a) n = 50, p = 0.5 (b) 0.0960

7 Completing the square (All boards)

$f(x) = x^2 - 6x + 3$.
(a) Write f(x) in the form $a(x + p)^2 + q$.
(b) State the minimum value of f(x) and the value of x at which it occurs.
(c) Solve f(x) = 0, giving surds in simplest form.

$f(x) = (x - 3)^2 - 6$
(b) minimum value -6 when x = 3
(c) $(x - 3)^2 = 6$, so $(x - 3)^2 = 6$
$x - 3 = \pm\sqrt{6}$

Answer: (a) $(x - 3)^2 - 6$ (b) minimum -6 at x = 3 (c) $x = 3 \pm \sqrt{6}$

8 Optimisation problems (All boards)

A window is a rectangle of width 2x m and height h m, topped by a semicircle of radius x m. Its perimeter is 50 m.
Show that its area is $A = 50x - (2 + \frac{\pi}{2})x^2$, and find the maximum area to 3 s.f.

Perimeter: $2x + 2h + \pi x = 50$, so $h = \frac{50 - 2x - \pi x}{2}$

$A = 2xh + \frac{1}{2}\pi x^2 = x(50 - 2x - \pi x) + \frac{1}{2}\pi x^2 = 50x - (2 + \frac{\pi}{2})x^2$

$\frac{dA}{dx} = 50 - (4 + \pi)x = 0$, so $x = \frac{50}{4 + \pi} = 7.00$

$\frac{d^2A}{dx^2} = -(4 + \pi) < 0$, so a maximum; A = 175

Answer: $A_{max} = 175\ m^2$ (x = 7.00)

9 Exponential equations: quadratic in b^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 12 \times 2^x + 32 = 0$.

Let $y = 2^x$. Then $y^2 - 12y + 32 = 0$
$(y - 4)(y - 8) = 0$
y = 4 or y = 8
$2^x = 4$ gives x = 2; $2^x = 8$ gives x = 3

Answer: x = 2 or x = 3

10 Product rule: stationary points (All boards)

$y = x \ln x$ for $x > 0$.

Find the exact coordinates of the stationary point and show that it is a minimum.

$\frac{dy}{dx} = \ln x + x \times \frac{1}{x} = \ln x + 1$

$\frac{dy}{dx} = 0$ gives $\ln x = -1$, so $x = e^{-1}$

$y = x^1 \ln x = e^{-1} \times (-\frac{1}{1}) = -e^{-1}$

$\frac{d^2y}{dx^2} = e > 0$ at this point, so it is a minimum

Answer: $(e^{-1}, -e^{-1})$, minimum