Day 125 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Smallest whole number solving an exponential inequality (All boards)

Find the smallest positive integer n such that

$(\frac{9}{10})^n < 0.1$.

Take logs: $n \ln \frac{9}{10} < \ln 0.1$

$\ln \frac{9}{10}$ is negative, so dividing reverses the inequality:

$n > \frac{\ln 0.1}{\ln \frac{9}{10}} = 21.85$, so $n = 22$

Answer: n = 22

2 Velocity, rest and total distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on the x-axis with velocity $v = (6t^2 - 42t + 72)$ m s^{-1} at time t s, $t \geq 0$, and starts at O. Find (a) the times when P is at instantaneous rest, (b) the total distance P travels in the first 6 s.

(a) $v = 6(t - 3)(t - 4) = 0$, so $t = 3$ or $t = 4$

(b) Displacement $x = \int v\,dt = 2t^3 - 21t^2 + 72t$

$x(3) = 81$, $x(4) = 80$, $x(6) = 108$

Direction changes at $t = 3$ and $t = 4$, so distance $= |81| + |80 - 81| + |108 - 80|$

Answer: (a) t = 3, t = 4 (b) 110 m

3 Dividing a line in a ratio (2D position vectors) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B have position vectors $\begin{pmatrix}-2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\5\end{pmatrix}$. The point P lies on AB with AP : PB = 3 : 1.

Find the position vector of P and the exact distance OP.

$\overrightarrow{OP} = \overrightarrow{OA} + \frac{3}{4}\overrightarrow{AB} = \begin{pmatrix}-2\\1\end{pmatrix} + \frac{3}{4}\begin{pmatrix}4\\4\end{pmatrix}$

$= \begin{pmatrix}1\\4\end{pmatrix}$

$OP = \sqrt{1^2 + 4^2} = \sqrt{17} = \sqrt{17}$

Answer: $\begin{pmatrix}1\\4\end{pmatrix}$, OP = $\sqrt{17}$

4 Newton's second law with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

A particle of mass 3 kg is acted on by forces $\mathbf{F_1} = (-4\mathbf{i} - 18\mathbf{j})$ N and $\mathbf{F_2}$. Its acceleration is $(-3\mathbf{i} - 2\mathbf{j})$ m s^{-2}. Find (a) $\mathbf{F_2}$, (b) the angle between $\mathbf{F_2}$ and $\mathbf{i}$, to 1 d.p.

$\mathbf{F} = m\mathbf{a}$: $\mathbf{F_1} + \mathbf{F_2} = 3(-3\mathbf{i} - 2\mathbf{j}) = -9\mathbf{i} - 6\mathbf{j}$

(a) $\mathbf{F_2} = (-9\mathbf{i} - 6\mathbf{j}) - (-4\mathbf{i} - 18\mathbf{j}) = -5\mathbf{i} + 12\mathbf{j}$

(b) $|\mathbf{F_2}| = \sqrt{(-5^2 + 12^2)} = 13$

$\cos\theta = -5 \div 13$, so $\theta = 112.6°$

Answer: (a) $\mathbf{F_2} = (-5\mathbf{i} + 12\mathbf{j})$ N (b) 112.6°

5 Factor theorem and remainder theorem (All boards)

$f(x) = x^3 + ax^2 + bx + 8$, where a and b are constants.

$(x - 4)$ is a factor of f(x). The remainder when f(x) is divided by $(x + 1)$ is 10.

(a) Find a and b. (b) Solve $f(x) = 0$.

$f(4) = 0$: $64 + 16a + 4b + 8 = 0$

$f(-1) = 10$: $-1 + a - b + 8 = 10$

Solving simultaneously: $a = -3$, $b = -6$

$f(x) = x^3 - 3x^2 - 6x + 8 = (x - 1)(x - 4)(x + 2)$

Answer: (a) a = −3, b = −6 (b) x = −2, 1, 4

6 Poisson over different intervals (CIE)

Customers enter a shop at random at an average of 3 per hour.
Find the probability that in a 20-minute period there are (a) zero events (b) at least 2.
(c) Find the probability that at least one occurs in each of two separate 20-minute periods.

$\lambda = 3 \times \frac{20}{60} = 1$, $X \sim Po(1)$

(a) $P(X = 0) = e^{-1} = 0.368$
(b) $P(X \geq 2) = 1 - P(X \leq 1) = 0.264$
(c) $(1 - 0.368)^2 = 0.400$
Answer: (a) 0.368 (b) 0.264 (c) 0.400

7 Exponential growth and decay modelling (All boards)

The number of fish in a lake is modelled by $N = 250\, e^{kt}$, where t is time in years.
After 4 years, N = 750.
Find (a) the exact value of k, (b) the time for N to reach 25000, to 3 s.f.

$250\, e^{4k} = 750$, so $e^{4k} = 3$ and $k = \frac{\ln 3}{4}$

$k = 0.275$ (3 s.f.)

$250\, e^{kt} = 25000$ gives $kt = \ln 100$, so $t = \frac{\ln 100}{k} = 16.8$ years

Answer: (a) $k = \frac{\ln 3}{4}$ (b) t = 16.8 years

8 Modulus inequalities (All boards)

Solve $|2x - 3| > |x - 3|$.

Square both sides: $(2x - 3)^2 > (x - 3)^2$
$((2x - 3) - (x - 3))((2x - 3) + (x - 3)) > 0$
$(x)(3x - 6) > 0$
Critical values x = 0 and x = 2; sketch the (positive or negative) quadratic.
Answer: x < 0 or x > 2

9 Partial fractions: repeated factor (All boards)

Express $\frac{2x^2 - 13x - 29}{(x - 4)(x + 3)^2}$ in the form $\frac{A}{(x - 4)} + \frac{B}{(x + 3)} + \frac{C}{(x + 3)^2}$, and find A, B and C.

$2x^2 - 13x - 29 \equiv A(x + 3)^2 + B(x - 4)(x + 3) + C(x - 4)$
$x = 4$: $-49 = A(7)^2$ so $A = -1$
$x = -3$: $28 = C(-7)$ so $C = -4$
Coefficient of x^2: $2 = A + B$ so $B = 3$
Answer: A = −1, B = 3, C = −4

10 Continuous random variable: finding k and E(X) (CIE)

The random variable X has probability density function $f(x) = kx$ for $0 \leq x \leq 2$, and 0 otherwise.
(a) Find k. (b) Find E(X). (c) Find $P(X > \frac{4}{3})$.

(a) $\int_0^2 f(x)\, dx = 1$ gives $2k = 1$, so $k = \frac{1}{2}$

(b) $E(X) = \int x\, f(x)\, dx = \frac{4}{3}$

(c) $P(X > \frac{4}{3}) = \int_{\frac{4}{3}}^2 f(x)\, dx = \frac{5}{9}$

Answer: (a) $k = \frac{1}{2}$ (b) $\frac{4}{3}$ (c) $\frac{5}{9}$