Day 121 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Arithmetic sequences: finding a and d (All boards)

The 5th term of an arithmetic sequence is 36 and the 13th term is 76.
Find the first term and common difference, and the sum of the first 20 terms.

$u_{13} - u_5 = 8\,d = 76 - 36$ so $d = 5$
$a + 4\,d = 36$ so $a = 16$
$S_{20} = \frac{20}{2}(2 \times 16 + 19 \times 5)$
Answer: a = 16, d = 5, $S_{20} = 1270$

2 Stratified sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A college has 135 students: 20 Arts, 40 Science and 75 Humanities.
A stratified sample of size 27 is taken.
Find the number of students sampled from each group.

Sampling fraction $= \frac{27}{135} = \frac{1}{5}$
Arts: $20 \times \frac{1}{5} = 4$
Science: $40 \times \frac{1}{5} = 8$
Humanities: $75 \times \frac{1}{5} = 15$
Answer: Arts 4, Science 8, Humanities 15

3 Change of sign to locate a root (All boards)

$f(x) = x^3 - 4x - 5$.
Show that the equation $f(x) = 0$ has a root α in the interval $[2, 3]$.

f(x) is continuous on the interval
$f(2) = -5.00 < 0$
$f(3) = 10.0 > 0$
A change of sign, so a root lies in [2, 3]
Answer: Sign change: f(2) = −5.00, f(3) = 10.0

4 Vertical motion under gravity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards at 9.8 m s^{-1} from a point h m above horizontal ground, and hits the ground 6 s later.
Find (a) h, (b) the greatest height above the ground. Take g = 9.8.

(a) Take upwards as positive: $s = -h$, $u = 9.8$, $a = -9.8$, $t = 6$
$-h = 9.8 \times 6 - 4.9 \times 6^2$, so $h = 117.6$
(b) At the highest point $v = 0$: $0 = 9.8^2 - 2 \times 9.8 \times s$
$s = 4.9$ m above the point of projection
Greatest height above the ground = 117.6 + 4.9
Answer: (a) 117.6 m (b) 122.5 m

5 Venn diagram with algebra (All boards)

The Venn diagram shows the numbers of 53 students in sets A and B.
(a) Find the value of x.
(b) A student in B is chosen at random. Find P(A).

(a) Total: $(2x + 4) + (x + 4) + 2x + 5 = 53$
$5x + 13 = 53$, so $x = 8$
(b) Regions: 20, 12, 16, 5
In B: 12 + 16 = 28, of which in A: 12
Answer: (a) x = 8 (b) $\frac{3}{7}$

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6 Circle touching an axis (diagram) (AQA, Edexcel, OCR, MEI, Eduqas)

The diagram shows a circle that touches the x-axis at A(−2, 0) and passes through B(3, 5).
Find the equation of the circle.

Touches the x-axis at A, so the centre is (−2, r) and the radius is r.

B on the circle: $(5)^2 + (5 - r)^2 = r^2$

$25 + 25 - 10r = 0$ so $r = 5$

Answer: $(x + 2)^2 + (y - 5)^2 = 25$

7 Catching up (delayed start) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes P at constant speed 16 m s^{-1}. 12 s later car B leaves P from rest, in the same direction, with acceleration 2 m s^{-2}. B overtakes A at Q. Find (a) the time B takes to reach Q, (b) PQ, (c) the speed of B at Q.

Let B take s seconds; A has then been moving for (s + 12) seconds.

B: $\frac{1}{2} \times 2 \times s^2 = 1\ s^2$ A: $16(s + 12)$

$1\ s^2 - 16\ s - 192 = 0$ so $s = 24$ ($s > 0$)

(b) $PQ = 1 \times 24^2 = 576$

(c) $v = 0 + at = 2 \times 24 = 48$

Answer: (a) 24 s (b) 576 m (c) 48 m s^{-1}

8 Greatest and least values using R-form (All boards)

$g(\theta) = \frac{1}{23 + 12 \sin\theta + 9 \cos\theta}$

(a) Write $12 \sin\theta + 9 \cos\theta$ in the form $R \sin(\theta + \alpha)$, with α acute.

(b) Find the greatest and least values of $g(\theta)$ and the smallest positive θ at which the greatest value occurs (1 d.p.).

$R = \sqrt{12^2 + 9^2} = 15$, $\alpha = \tan^{-1}\frac{9}{12} = 36.9°$

Denominator $= 23 + 15 \sin(\theta + 36.9°)$, ranging from 8 to 38

Greatest $g = \frac{1}{8}$ (denominator least), least $g = \frac{1}{38}$

Least denominator when $\sin(\theta + 36.9°) = -1$: $\theta + 36.9° = 270°$

$\theta = 233.1°$

Answer: R = 15, α = 36.9°; greatest $\frac{1}{8}$, least $\frac{1}{38}$; θ = 233.1°

9 Recurrence relations: limits (All boards)

$u_1 = 0$ and $u_{n+1} = \frac{1}{3} u_n + 10$.

(a) Find the limit L of the sequence as $n \to \infty$.

(b) Find the least n for which $|u_n - L| < 0.1$.

(a) $L = \frac{1}{3} L + 10$ gives $L = \frac{10}{1 - \frac{1}{3}} = 15$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = (-15) \times \left(\frac{1}{3}\right)^{n-1}$

$|-15| \times \left(\frac{1}{3}\right)^{n-1} < \frac{1}{10}$

Check: n = 5 gives 0.18519, n = 6 gives 0.06173

Answer: (a) L = 15 (b) n = 6

10 Collinear points and ratios (3D) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A, B and C have position vectors $\begin{pmatrix}-2\\4\\-4\end{pmatrix}$, $\begin{pmatrix}-2\\3\\-6\end{pmatrix}$ and $\begin{pmatrix}-2\\k\\-2\end{pmatrix}$

relative to O, where k is a constant. Given that A, B and C are collinear, find k and the ratio AB : BC.

AB = $\begin{pmatrix}0\\-1\\-2\end{pmatrix}$, and AC = $\begin{pmatrix}0\\k-4\\2\end{pmatrix}$

AC = λAB: component 2 gives $1 = -1\ \lambda$, so $\lambda = -1$

Hence $k = 4 + (-1) \times (-1) = 5$

AC = −1 AB, so BC = −2 AB: AB : BC = 1 : 2

Answer: k = 5, AB : BC = 1 : 2