Day 12 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Two balls meeting (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is dropped from rest 98 m above the ground. At the same instant ball B is projected upwards at 24.5 m s^{-1} from the ground directly below A. Find (a) the time when they collide, (b) the height of the collision. Take g = 9.8.

Let the time be t. Distance fallen by A: $\frac{1}{2}gt^2 = 4.9t^2$
Height of B above the ground: $24.5\,t - 4.9t^2$
They meet when the distances add up to 98: $4.9t^2 + (24.5\,t - 4.9t^2) = 98$
so $24.5\,t = 98$, $t = 4$
Height of B = $24.5 \times 4 - 4.9 \times 4^2 = 19.6$
Answer: (a) 4 s (b) 19.6 m

2 Binomial probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(13, 0.2). Find, to 3 significant figures:
(a) $P(2 \leq X \leq 4)$ (b) $P(X > 2)$

(a) $P(X \leq 4) - P(X \leq 1) = 0.901 - 0.234 = 0.667$
(b) $1 - P(X \leq 2) = 1 - 0.502 = 0.498$
Answer: (a) 0.667 (b) 0.498

3 Lift and cable tension (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 250 kg carries a person of mass 60 kg. The cable tension is 2852 N and the lift is accelerating. Find (a) the magnitude and direction of the acceleration, (b) the force of the floor on the person. Take g = 9.8.

Whole system (mass 310 kg), taking upwards as positive: $T - 310\,g = 310\,a$
$2852 - 3038 = 310\,a$, so $a = -0.6$
(a) a is negative, so the acceleration is 0.6 m s^{-2} downwards
(b) Person: $R - 60\,g = 60\,a$, so $R = 60(9.8 - 0.6)$
Answer: (a) 0.6 m s^{-2} downwards (b) 552 N

4 Proof by deduction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove that $n^3 - n$ is divisible by 6 for every positive integer n.

$n^3 - n = n(n-1)(n+1) = (n-1)\,n\,(n+1)$
This is a product of three consecutive integers.
One of any three consecutive integers is a multiple of 3, and at least one is even.
So the product is a multiple of 2 and of 3, hence of 6.
Answer: $(n-1)n(n+1)$ is a multiple of 2 and 3, so of 6

5 Perpendicular vectors and a constant (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The vectors $\begin{pmatrix} k \\ k \\ -2 \end{pmatrix}$ and $\begin{pmatrix} k \\ 3 \\ 2 \end{pmatrix}$ are perpendicular.

Find the possible values of k and, for the larger value, the exact length of the first vector.

Scalar product = 0: $k^2 + 3k - 4 = 0$
$(k-1)(k+4) = 0$
$k = -4$ or $k = 1$. For $k = 1$ the first vector is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, with length $\sqrt{6} = \sqrt{6}$

Answer: $k = -4$ or 1; length $\sqrt{6}$

6 Inverse of a quadratic function (All boards)

$f(x) = 3x^2 - 5$, $x \geq 0$.

Find $f^{-1}(x)$ and state its domain.

$f(x) = 3x^2 - 5$

Let $y = f(x)$: $x^2 = \frac{y + 5}{3}$

$x \geq 0$, so take the positive root: $x = \sqrt{\frac{y + 5}{3}}$

Answer: $f^{-1}(x) = \sqrt{\frac{x + 5}{3}}$, $x \geq -5$

7 Parametric differentiation with trigonometry (All boards)

A curve has parametric equations $x = (t - \sin t)$, $y = (1 - \cos t)$.

Find $\frac{dy}{dx}$ in terms of t and the exact equation of the tangent at the point where $t = \frac{\pi}{2}$, in the form $y = mx + c$.

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = \frac{\sin t}{1 - \cos t}$

$t = \frac{\pi}{2}$: gradient $m = 1$, point $(\frac{\pi - 2}{2}, 1)$

$c = y - mx = \frac{4 - \pi}{2}$

Answer: $\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$; $y = x + \frac{4 - \pi}{2}$

8 Recurrence relations: unknown first term (All boards)

A sequence is defined by $u_1 = k$ and $u_{n+1} = -u_n + 6$.

Given that $u_3 = 7$, find (a) the value of k, (b) $\sum_{r=1}^{4} u_r$.

$u_2 = -1\,k + 6$, $u_3 = (-1)(-1\,k + 6) + 6$

$k = 7$ so $k = 7$

Terms: 7, −1, 7, −1

Sum = 12

Answer: (a) k = 7 (b) 12

9 Integration by parts (All boards)

Use integration by parts twice to find the exact value of

$\int_0^3 x^2 e^x \, dx$.

$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$

$\int x e^x \, dx = x e^x - e^x$, so $\int x^2 e^x \, dx = e^x(x^2 - 2x + 2)$

$= 5e^3 - 2$

Answer: $5e^3 - 2$

10 Poisson over different intervals (CIE)

Faults are reported at random at an average of 2 per hour.

Find the probability that in a 45-minute period there are (a) zero events (b) at least 2.

(c) Find the probability that at least one occurs in each of two separate 45-minute periods.

$\lambda = 2 \times \frac{45}{60} = 1.5$, $X \sim Po(1.5)$

(a) $P(X = 0) = e^{-1.5} = 0.223$

(b) $P(X \geq 2) = 1 - P(X \leq 1) = 0.442$

(c) $(1 - 0.223)^2 = 0.604$

Answer: (a) 0.223 (b) 0.442 (c) 0.604