Day 115 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Change of sign to locate a root (All boards)

$f(x) = x^3 - 2x - 1$.
Show that the equation $f(x) = 0$ has a root α in the interval $[1, 2]$.

$f(x)$ is continuous on the interval
$f(1) = -2.00 < 0$
$f(2) = 3.00 > 0$
A change of sign, so a root lies in $[1, 2]$
Answer: Sign change: $f(1) = -2.00$, $f(2) = 3.00$

2 Particle supported by two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 2 kg particle hangs in equilibrium from two light strings, as shown. Find the tension in each string (T_1 in the string at angle α). Take $g = 9.8$.

$\sin\alpha = 0.8$, $\cos\alpha = 0.6$, $\sin\beta = \frac{12}{13}$, $\cos\beta = \frac{5}{13}$
Horizontal: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_2 = 1.56\, T_1$
Vertical: $T_1 \sin\alpha + T_2 \sin\beta = 2g = 19.6$
$T_1(0.8 + 1.44) = 19.6$
$T_1 = 8.75$, $T_2 = 13.7$
Answer: $T_1 = 8.75$ N, $T_2 = 13.65$ N

3 Differentiating sec, cosec, tan and cot (All boards)

$y = \cot(x)$.
Find $\frac{dy}{dx}$ and its exact value at $x = \frac{\pi}{3}$.

$\frac{dy}{dx} = -\operatorname{cosec}^2(x)$
At $x = \frac{\pi}{3}$ the angle $x = \frac{\pi}{3}$, so the gradient is $-\frac{4}{3}$
Answer: $\frac{dy}{dx} = -\operatorname{cosec}^2(x)$; $-\frac{4}{3}$

4 Beam on two supports (AQA, Edexcel, OCR, MEI)

A uniform beam AB rests on supports at C and D, with a particle attached at B. Find the reactions at C and D. Take $g = 9.8$.

Moments about C: $R_D \times 3.5 = 30g \times 1 + 5g \times 4$
$R_D = 140$
Resolving vertically: $R_C + R_D = 35g = 343$
$R_C = 343 - 140$
Answer: $R_C = 203$ N, $R_D = 140$ N

5 Cubic inequalities (All boards)

Given that $(x + 5)$ is a factor of $f(x) = x^3 + 5x^2 - 16x - 80$, solve $f(x) > 0$.

Factorise: $f(x) = (x + 5)(x + 4)(x - 4)$
Roots at $x = -5, -4, 4$
Sketch the cubic (positive x^3 coefficient: negative, positive, negative, positive)
Read off where $f(x) > 0$.
Answer: $-5 < x < -4$ or $x > 4$

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6 Integration by substitution (All boards)

Use the substitution $u = x^2$ to find the exact value of

$\int_0^1 x\, e^{2x^2}\, dx$.

$du = 2x\, dx$, so $x\, dx = \frac{1}{2}du$; limits: $u = 0$ to $u = 1$

$I = \frac{1}{2}\int_0^1 e^{2u}\, du = \frac{1}{2}[\frac{e^{2u}}{2}]$

$= \frac{e^2 - 1}{4}$

Answer: $\mathbf{\frac{e^2 - 1}{4}}$

7 Factor theorem and remainder theorem (All boards)

$f(x) = x^3 + ax^2 + bx - 9$, where a and b are constants.
$(x - 3)$ is a factor of $f(x)$. The remainder when $f(x)$ is divided by $(x + 2)$ is 5.
(a) Find a and b. (b) Solve $f(x) = 0$.

$f(3) = 0$: $27 + 9a + 3b - 9 = 0$
$f(-2) = 5$: $-8 + 4a - 2b - 9 = 5$
Solving simultaneously: $a = 1$, $b = -9$
$f(x) = x^3 + x^2 - 9x - 9 = (x + 1)(x + 3)(x - 3)$
Answer: (a) a = 1, b = −9 (b) x = −3, −1, 3

8 Histograms (All boards)

The histogram shows the distances thrown, d (in metres), of a sample.
(a) How many were in the class $0 \leq d < 5$?
(b) Estimate how many had $30 < d < 55$.

(a) Frequency = frequency density × width = 7 × 5 = 35
(b) 30 to 40: 4 × 10 = 40
40 to 55: 5 × 15 = 75
Total = 115
Answer: (a) 35 (b) 115

9 Coding mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 25 values of x, $\Sigma(x - 40) = 125$ and $\Sigma(x - 40)^2 = 850$.
Find the mean and standard deviation of x.

Let $y = x - 40$: $\bar{y} = \frac{125}{25} = 5$
Mean of $x = 5 + 40 = 45$
Variance = $\frac{850}{25} - 5^2 = 9$
Subtracting a constant does not change spread, so sd = 3
Answer: mean 45, sd 3

10 Recurrence relations: limits (All boards)

$u_1 = 0$ and $u_{n+1} = -\frac{1}{2} u_n + 30$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.1$.

(a) $L = -\frac{1}{2} L + 30$ gives $L = \frac{30}{1 - (-\frac{1}{2})} = 20$

(b) $u_n - L = (u_1 - L)\, r^{n-1} = (-20) \times (-\frac{1}{2})^{n-1}$

$|-20| \times \frac{1}{2}^{n-1} < \frac{1}{10}$

Check: n = 8 gives 0.15625, n = 9 gives 0.07812

Answer: (a) L = 20 (b) n = 9