Day 103 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Work-energy on a rough slope (CIE)

A 3 kg block is released from rest on a rough plane inclined at α, $\tan\alpha = \frac{4}{3}$, $\mu = 0.5$, and slides 10 m down a line of greatest slope. Use work and energy to find (a) the work done against friction, (b) its speed after 10 m. Take $g = 10$.

$R = 3g\cos\alpha = 18$ N, $F = \mu R = 9$ N
(a) Work done against friction $= F \times d = 9 \times 10$
(b) Loss of PE $= 3g \times 10\sin\alpha = 240$ J
Work-energy: $\frac{1}{2}\times 3\times v^2 = 240 - 90 = 150$
$v^2 = 100$
Answer: (a) 90 J (b) 10 m s^{-1}

2 Proof by deduction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove that $n^2 + n + 5$ is odd for every positive integer n.

$n^2 + n + 5 = n(n + 1) + 5$
$n(n + 1)$ is the product of consecutive integers, so it is even.
Even + 5 (odd) = odd.
Answer: $n(n + 1)$ is even, so $n^2 + n + 5$ is odd

3 Stratified sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A college has 248 students: 64 Arts, 112 Science and 72 Humanities.
A stratified sample of size 31 is taken.
Find the number of students sampled from each group.

Sampling fraction $= \frac{31}{248} = \frac{1}{8}$

Arts: $64 \times \frac{1}{8} = 8$

Science: $112 \times \frac{1}{8} = 14$

Humanities: $72 \times \frac{1}{8} = 9$

Answer: Arts 8, Science 14, Humanities 9

4 Change of sign to locate a root (All boards)

$f(x) = e^x - x - 4$.
Show that the equation $f(x) = 0$ has a root α in the interval $[1, 2]$.

$f(x)$ is continuous on the interval
$f(1) = -2.28 < 0$
$f(2) = 1.39 > 0$
A change of sign, so a root lies in $[1, 2]$
Answer: Sign change: $f(1) = -2.28$, $f(2) = 1.39$

5 Line and curve intersection (diagram) (All boards)

The diagram shows the curve $y = x^2 - x - 7$ and the line $y = -x - 3$, meeting at A and B.
(a) Find the coordinates of A and B.
(b) Find the coordinates of the midpoint of AB.

$x^2 - x - 7 = -x - 3$: $x^2 - 4 = 0$
$(x + 2)(x - 2) = 0$, $x = -2$ or 2
A$(-2, -1)$, B$(2, -5)$
Midpoint: average the coordinates
Answer: (a) A$(-2, -1)$, B$(2, -5)$ (b) $(0, -3)$

6 Implicit differentiation: tangent to a curve (All boards)

A curve has equation $x^2 + xy + 2x - 2y = -4$.
Show that P(0, 2) lies on the curve and find the equation of the tangent at P in the form ax + by + c = 0 with integer a, b, c.

Differentiate (product rule on the xy term): $(2x + y + 2) + (x - 2)\frac{dy}{dx} = 0$

At P(0, 2): $4 + (-2)\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = 2$

Tangent: $y - 2 = 2(x + 0)$, i.e. $2x - y + 2 = 0$

Answer: $2x - y + 2 = 0$

7 Circle and line: chord length (AQA, Edexcel, OCR, MEI, Eduqas)

The line $3x + 4y = -2$ meets the circle $(x - 1)^2 + (y + 3)^2 = 49$ at the points A and B. Find the exact length of AB.

Centre (1, −3), radius 7. Distance from centre to line:

$d = \frac{|-7|}{5} = \frac{7}{5}$

$\text{Half-chord}^2 = 49 - (\frac{7}{5})^2 = \frac{1176}{25}$

$AB = 2\sqrt{\frac{1176}{25}}$

Answer: $\frac{140\sqrt{6}}{25}$

8 Catching up (delayed start) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes P at constant speed 6 m s^{-1}. 9 s later car B leaves P from rest, in the same direction, with acceleration 1 m s^{-2}. B overtakes A at Q. Find (a) the time B takes to reach Q, (b) PQ, (c) the speed of B at Q.

Let B take s seconds; A has then been moving for (s + 9) seconds.
B: $\frac{1}{2}\times 1\times s^2 = 0.5\,s^2$ A: $6(s + 9)$
$0.5\,s^2 - 6\,s - 54 = 0$ so $s = 18$ ($s > 0$)
(b) $PQ = 0.5\times 18^2 = 162$
(c) $v = 0 + at = 1\times 18 = 18$
Answer: (a) 18 s (b) 162 m (c) 18 m s^{-1}

9 Reducing a relationship to a straight line (graph) (All boards)

The diagram shows the graph of ln y against ln x for $y = A x^n$.
Find the value of n and the exact value of A.

$\ln y = \ln A + n \ln x$: gradient n, intercept ln A

$n = \frac{-2 - 0}{4 - 2} = -1$

At (2, 0): $\ln A = 0 - (-1) \times 2 = 2$, so $A = e^2$

Answer: $n = -1$, $A = e^2$

10 Smallest number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The probability that a driver passes the test is 0.25 on each independent trial.
Find the least number of trials needed so that the probability of this happening at least once is greater than 0.995.

P(at least one) = $1 - 0.75^n > 0.995$
$0.75^n < 0.005$
$n > \frac{\log 0.005}{\log 0.75} = 18.4$
Least integer n = 19
Answer: 19