Day 10 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Fixed point iteration (All boards)

The equation $e^x = 4x + 4$ has a root α. Use the iteration $x_{n+1} = \ln(4x + 4)$ with $x_0 = 3$ to find x_1, x_2 and x_3 to 4 d.p., and hence the value of α to 2 d.p. (continue until the 2 d.p. value is settled).

$x_1 = \ln(4 \times 3 + 4) = 2.7726$

$x_2 = 2.7141$, $x_3 = 2.6984$

Continuing, the values settle at 2.6926, so $\alpha = 2.69$

Answer: $x_1 = 2.7726$, $x_2 = 2.7141$, $x_3 = 2.6984$; $\alpha = 2.69$

2 Conservation of momentum (CIE)

Particles A (8 kg) and B (2 kg) move in the same direction on a smooth horizontal line at 8 m s^{-1} and 7 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):

$8\times8 + 2\times7 = 10v$, so $v = 7.8$

KE before = $\frac{1}{2}\times8\times8^2 + \frac{1}{2}\times2\times7^2 = 305$

KE after = $\frac{1}{2}\times10\times7.8^2 = 304.2$

Loss = $305 - 304.2$

Answer: (a) 7.8 m s^{-1} (b) 0.8 J

3 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has $P(X = x) = k(x + 3)$ for $x = 2, 3, 4, 5$.

(a) Show that $k = \frac{1}{26}$. (b) Find E(X) and Var(X).

(c) Find $P(X \geq 3)$.

(a) $\Sigma P(X = x) = 1$: $5 + 6 + 7 + 8 = 26$, so $26k = 1$

(b) $E(X) = 2 \times \frac{5}{26} + 3 \times \frac{3}{13} + 4 \times \frac{7}{26} + 5 \times \frac{4}{13} = \frac{48}{13}$

$E(X^2) = \frac{193}{13}$, so $Var(X) = \frac{193}{13} - (\frac{48}{13})^2 = \frac{205}{169}$

(c) $P(X \geq 3) = \frac{3}{13} + \frac{7}{26} + \frac{4}{13} = \frac{21}{26}$

Answer: (b) $E(X) = \frac{48}{13}$, $Var(X) = \frac{205}{169}$ (c) $\frac{21}{26}$

4 Lift and cable tension (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 500 kg carries a person of mass 50 kg. The cable tension is 5170 N and the lift is accelerating. Find (a) the magnitude and direction of the acceleration, (b) the force of the floor on the person. Take g = 9.8.

Whole system (mass 550 kg), taking upwards as positive: $T - 550g = 550a$

$5170 - 5390 = 550a$, so $a = -0.4$

(a) a is negative, so the acceleration is 0.4 m s^{-2} downwards

(b) Person: $R - 50g = 50a$, so $R = 50(9.8 - 0.4)$

Answer: (a) 0.4 m s^{-2} downwards (b) 470 N

5 Smallest number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The probability that a ticket wins a prize is 0.05 on each independent trial.

Find the least number of trials needed so that the probability of this happening at least once is greater than 0.995.

$P(\text{at least one}) = 1 - 0.95^n > 0.995$

$0.95^n < 0.005$

$n > \frac{\log 0.005}{\log 0.95} = 103$

Least integer $n = 104$

Answer: 104

6 Quadratic from its graph (All boards)

The diagram shows the curve with equation y = f(x), where f is a quadratic.
The turning point is marked, and the curve crosses the y-axis as shown.
Find f(x), giving it in the form $ax^2 + bx + c$.

Turning point (1, −5): $f(x) = a(x - 1)^2 - 5$
$f(0) = -8$: $a(1)^2 - 5 = -8$
$a = -3$, then expand
Answer: $f(x) = -3x^2 + 6x - 8$

7 Partial fractions and integration (All boards)

(a) Express $\frac{-4x - 14}{(x + 1)(x + 6)}$ in partial fractions.

(b) Hence find $\int_0^6 \frac{-4x - 14}{(x + 1)(x + 6)}\,dx$, giving the answer in the form ln k.

$-4x - 14 \equiv A(x + 6) + B(x + 1)$: $x = -6$ gives $A = -2$; $x = -1$ gives $B = -2$
$I = [-2\ln(x + 1) - 2\ln(x + 6)]_0^6$
$= -2\ln\frac{7}{1} - 2\ln\frac{12}{6}$
$= \ln\frac{1}{196}$
Answer: (a) $\frac{-2}{x + 1} - \frac{2}{x + 6}$; (b) $\ln\frac{1}{196}$

8 Periodic sequences (All boards)

A sequence is defined by $u_1 = 2$ and $u_{n+1} = 1 - \frac{1}{u_n}$.

(a) Find u_2 and u_3.

(b) Find u_{99} and $\sum_{r=1}^{99} u_r$.

$u_2 = \frac{1}{2}$, $u_3 = -1$, $u_4 = 2$
$u_4 = u_1$, so the sequence repeats with period 3.
$99 = 3 \times 33 + 0$, so $u_{99} = u_3 = -1$
Sum of one period = $\frac{3}{2}$; 33 full periods plus 0 extra term(s)
Answer: (a) $\frac{1}{2}$, -1 (b) $u_{99} = -1$, sum = $\frac{99}{2}$

9 Implicit differentiation: products and cubes (All boards)

A curve has equation $3x^2y - y^3 + 4x = -6$.

Find $\frac{dy}{dx}$ in terms of x and y, and the gradient of the normal at (−1, 2).

Differentiate: $6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} + 4 = 0$
$\frac{dy}{dx} = -\frac{6xy + 4}{3x^2 - 3y^2}$
At (−1, 2): $\frac{dy}{dx} = -\frac{8}{9}$, so the normal has gradient $\frac{9}{8}$
Answer: $\frac{dy}{dx} = -\frac{6xy + 4}{3x^2 - 3y^2}$; normal gradient $\frac{9}{8}$

10 Partial fractions and binomial expansion (All boards)

$f(x) = \frac{10x - 6}{(1 - x)(1 - 2x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) up to x^2.

(c) State the range of x for which the expansion is valid.

(a) $f(x) = \frac{A}{(1 - x)} + \frac{B}{(1 - 2x)}$: $A = -4$, $B = -2$

(b) $-4(1 - x)^{-1} = -4(1 + x + 1x^2 ...)$

Add the series for both fractions.

(c) Need $|x| < 1$ and $|2x| < 1$

Answer: (a) $\frac{-4}{(1 - x)} + \frac{-2}{(1 - 2x)}$ **(b)** $-6 - 8x - 12x^2$ **(c)** $|x| < \frac{1}{2}$