Day 315 **S Tier 10-a-day: Model Solutions**

1 Capture and recapture

20 beetles in a field are caught, marked and put back.
Later, a second sample of 40 beetles is caught. 4 of these are marked.
(a) Work out an estimate for the number of beetles in a field.
(b) State one assumption you have made.

(a) The proportion marked in the second sample is 4 out of 40

This should match the proportion marked in the whole population: $\frac{4}{40} = \frac{20}{N}$

$N = 20 \times 40 \div 4 = 200$

(b) No beetles are born, die or leave between the samples, and the beetles mix evenly

Answer: (a) 200 (b) The population stays the same between samples

2 Fractions of fractions

Gareth cycles $\frac{5}{7}$ of a route before lunch.

After lunch he cycles $\frac{5}{9}$ of the remaining distance.

The distance he cycles after lunch is 30 km.
How long is the route, in km?

The second amount is $\frac{5}{9}$ of $\frac{2}{7}$, which is $\frac{10}{63}$ of the total

$\frac{10}{63}$ of the total is 30

Total = $30 \div \frac{10}{63} = 189$

Answer: 189 km

3 Exchange rates

Kira changes £1000 into dollars at £1 = $1.15.
Kira spends $170 and then changes the rest back into pounds at £1 = $1.6.
How many pounds does Kira get back?

£1000 × 1.15 = $1150
1150 − 170 = $980
980 ÷ 1.6 = £612.50
Answer: £612.50

4 Perimeter and area of a running track

The diagram shows a running track made from a rectangle and two semicircles.
(a) Work out the total length of the boundary of the track.
(b) Work out the area enclosed by the track. Give your answers in terms of π.
The diagram is not drawn accurately.

(a) The two semicircles make one full circle of radius 20: circumference = $2 \times \pi \times 20 = 40\pi$
The two straight sides: $2 \times 100 = 200$
Boundary = $(200 + 40\pi)$ m

(b) The rectangle is 100 m by $2 \times 20 = 40$ m, so its area is $100 \times 40 = 4000$. Circle: $\pi \times 20^2 = 400\pi$
Area = $(4000 + 400\pi)$ m^2

Answer: (a) $(200 + 40\pi)$ m (b) $(4000 + 400\pi)$ m^2

5 Expanding three brackets

Expand and simplify $(2x + 5)(x - 3)(x + 1)$

Expand the first two brackets: $(x - 3)(x + 1) = x^2 - 2x - 3$
Multiply by $(2x + 5)$: $2x(x^2 - 2x - 3) + 5(x^2 - 2x - 3)$
$= 2x^3 + x^2 - 16x - 15$
Answer: $2x^3 + x^2 - 16x - 15$

Day 315 **S Tier 10-a-day: Model Solutions**

6 Density of an alloy

An alloy is made by mixing 60 cm^3 of tin with 100 cm^3 of lead.
The density of tin is 7.3 g/cm^3. The density of lead is 11.3 g/cm^3.
Work out the density of the alloy.

Mass of tin = $60 \times 7.3 = 438$ g
Mass of lead = $100 \times 11.3 = 1130$ g
Total mass = 1568 g, total volume = 160 cm^3
Density = $1568 \div 160 = 9.8$ g/cm^3
Answer: 9.8 g/cm^3

7 Tangent to a circle: proof

Show that the line $y = -x - 14$ is a tangent to the circle $x^2 + y^2 = 98$.
Find the coordinates of the point where the line touches the circle.

Substitute $y = -x - 14$ into $x^2 + y^2 = 98$: $x^2 + (-x - 14)^2 = 98$
Expand: $2x^2 + 28x + 98 = 0$
$b^2 - 4ac = 28^2 - 4 \times 2 \times 98 = 784 - 784 = 0$
$b^2 - 4ac = 0$, so there is exactly one point of intersection: the line is a tangent
$x = \frac{-28}{2 \times 2} = -7$, and $y = -(-7) - 14 = -7$. The point is $(-7, -7)$
Answer: $(-7, -7)$

8 Rationalising the denominator

Write $\frac{5 + \sqrt{3}}{3 + \sqrt{3}}$ in the form $p + q\sqrt{3}$
where p and q are rational numbers.

Multiply top and bottom by $(3 - \sqrt{3})$:
Numerator: $(5 + \sqrt{3})(3 - \sqrt{3}) = 12 - 2\sqrt{3}$
Denominator: $9 - 3 = 6$
$= \frac{6 - \sqrt{3}}{3}$
Answer: $\frac{6 - \sqrt{3}}{3}$

9 Functions that are their own inverse

$f(x) = \frac{-5x - 4}{3x + 5}$ where $x \neq -\frac{5}{3}$
(a) Show that ff(x) = x.
(b) Write down $f^{-1}(x)$.

(a) Replace x in f(x) by f(x), then multiply the top and the bottom by (3x + 5)
$ff(x) = \frac{(-5)(-5x - 4) + (-4)(3x + 5)}{3(-5x - 4) - (-5)(3x + 5)}$
Top = $25x + 20 - 12x - 20 = 13x$
Bottom = $-15x - 12 + 15x + 25 = 13$
$ff(x) = \frac{13x}{13} = x$

(b) ff(x) = x, so f is its own inverse: $f^{-1}(x) = \frac{-5x - 4}{3x + 5}$

Answer: (b) $f^{-1}(x) = \frac{-5x - 4}{3x + 5}$

Day 315 **S Tier 10-a-day: Model Solutions**

10 Vectors: intersecting lines

OAB is a triangle. OA = $\mathbf{a}$ and OB = $\mathbf{b}$.
M is on OA with OM : MA = 3 : 2. N is on OB with ON : NB = 2 : 3. AN and BM intersect at X.
(a) Find BM in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Find OX in terms of $\mathbf{a}$ and $\mathbf{b}$.
The diagram is not drawn accurately.

(a) OM = $\frac{3}{5}\mathbf{a}$, so BM = $-\mathbf{b} + \frac{3}{5}\mathbf{a} = \frac{3}{5}\mathbf{a} - \mathbf{b}$

(b) X is on BM: BX = s BM, so OX = OB + s BM = $\frac{3}{5}s\mathbf{a} + (1 - s)\mathbf{b}$

X is on AN: OX = OA + t AN = OA + t(ON − OA) = $(1 - t)\mathbf{a} + \frac{2}{5}t\mathbf{b}$

Equate: $\frac{3}{5}s = 1 - t$ and $1 - s = \frac{2}{5}t$

Solving: $s = \frac{15}{19}$ and $t = \frac{10}{19}$

OX = $\frac{9}{19}\mathbf{a} + \frac{4}{19}\mathbf{b}$

Answer: (a) $\frac{3}{5}\mathbf{a} - \mathbf{b}$ (b) OX = $\frac{9}{19}\mathbf{a} + \frac{4}{19}\mathbf{b}$