Day 214 **S Tier 10-a-day: Model Solutions**

1 Completing the square

(a) Write $4x^2 - 16x + 7$ in the form $a(x + p)^2 + q$

(b) Write down the coordinates of the turning point of the curve $y = 4x^2 - 16x + 7$

(a) Take out the factor 4 from the x terms: $4(x^2 - 4x) + 7$

$x^2 - 4x = (x - 2)^2 - 4$, so $4x^2 - 16x + 7 = 4[(x - 2)^2 - 4] + 7$

$= 4(x - 2)^2 - 9$

(b) The minimum is when $x - 2 = 0$, so $x = 2$ and $y = -9$

Answer: (a) $4(x - 2)^2 - 9$ (b) $(2, -9)$

2 Cone made from a sector

A sector of a circle has radius 24 cm and angle 150°.

The sector is made into a cone by joining the two straight edges together.

(a) Work out the radius of the base of the cone.

(b) Work out the exact volume of the cone. Give your answer in terms of π, as a surd if necessary.

(a) Arc length of the sector = $\frac{150}{360} \times 2\pi \times 24 = 20\pi$ cm

This is the circumference of the base: $2\pi r = 20\pi$, so $r = 10$ cm

(b) Slant height = 24 cm, so height$^2 = 24^2 - 10^2 = 476$, height = $2\sqrt{119}$ cm

Volume = $\frac{1}{3} \times \pi \times 10^2 \times 2\sqrt{119} = \frac{200\sqrt{119}}{3}\pi$ cm^3

Answer: (a) 10 cm (b) $\frac{200\sqrt{119}}{3}\pi$ cm^3

3 Average speed

A train travels the first half of a journey at 30 km/h.

It travels the second half of the same journey at 60 km/h.

Work out the average speed for the whole journey.

Let each half of the journey be 60 km (a common multiple of 30 and 60)

Time for first half = 60 ÷ 30 = 2 hours

Time for second half = 60 ÷ 60 = 1 hours

Average speed = total distance ÷ total time = 120 ÷ 3 = 40 km/h

Answer: 40 km/h

4 Iteration

Use the iteration formula $x_{n+1} = 2 + \frac{2}{x_n}$ with $x_0 = 4$ (Calculator)

to work out x_1, x_2 and x_3.

Give each answer correct to 3 decimal places.

$x_1 = 2 + \frac{2}{4} = 2.50000...$

$x_2 = 2 + \frac{2}{x_1} = 2.80000...$

$x_3 = 2 + \frac{2}{x_2} = 2.71429...$

Rounded: $x_1 = 2.500$, $x_2 = 2.800$, $x_3 = 2.714$

Answer: $x_1 = 2.500$, $x_2 = 2.800$, $x_3 = 2.714$

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5 Tangents and circle theorems

PA and PB are tangents to a circle, centre O. C is a point on the circle. Angle ACB = 52°.
(a) Work out the size of angle AOB.
(b) Work out the size of angle APB.
The diagram is not drawn accurately.

(a) The angle at the centre is twice the angle at the circumference: 2 × 52 = 104°
(b) Angles OAP and OBP are 90° (a tangent meets a radius at 90°)
Angle APB = 360 − 90 − 90 − 104 = 76°
Answer: (a) 104° (b) 76°

6 Ratio and algebra

In a coach, the ratio of the number of adults to the number of children is 7 : 3.
Then 12 adults are removed and 2 more children are added.
The ratio is now 8 : 7.
How many children were there at the start?

Let the numbers at the start be 7k and 3k
(7k − 12) : (3k + 2) = 8 : 7, so 7(7k − 12) = 8(3k + 2)
49k − 84 = 24k + 16
25k = 100, so k = 4
Number of children at the start: 3 × 4 = 12
Answer: 12

7 Fractional indices

Given that x > 0, solve $x^{-\frac{3}{2}} = \frac{1}{1331}$

Raise both sides to the power $-\frac{2}{3}$

$x = (\frac{1}{1331})^{-\frac{2}{3}}$

x = 121
Answer: x = 121

8 Probability and quadratic equations

A bag contains some counters. 4 of them are green and the rest are white.
Two counters are taken at random without replacement.

The probability that both are green is $\frac{2}{7}$.

Work out how many counters are white.

Let there be n counters altogether

P(both green) = $\frac{4}{n} \times \frac{3}{n-1} = \frac{12}{n(n-1)}$

$\frac{12}{n(n-1)} = \frac{2}{7}$, so n(n − 1) = 42, giving $n^2 - n - 42 = 0$

(n − 7)(n + 6) = 0, so n = 7 (n cannot be negative)
Number of counters that are white = 7 − 4 = 3
Answer: 3

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9 Vector proof: collinear points

OA = $\mathbf{a}$ and OB = $\mathbf{b}$.
OP = $4\mathbf{b}$, OQ = $4\mathbf{a} + 10\mathbf{b}$ and OR = $6\mathbf{a} + 13\mathbf{b}$.
(a) Show that P, Q and R lie on a straight line.
(b) Work out the ratio PQ : QR in its simplest form.

(a) PQ = OQ − OP = $4\mathbf{a} + 6\mathbf{b}$
QR = OR − OQ = $2\mathbf{a} + 3\mathbf{b}$

QR = $\frac{1}{2}$ PQ, so PQ and QR are parallel, and they share the point Q

So P, Q and R lie on a straight line
(b) PQ : QR = 2 : 1
Answer: (b) 2 : 1

10 Nested surds

Show that $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$

Square the right-hand side: $(1 + \sqrt{3})^2 = 1 + 2\sqrt{3} + 3$
$= 4 + 2\sqrt{3}$
So $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$
Answer: $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$