Day 209 **S Tier 10-a-day: Model Solutions**

1 Capture and recapture

20 fish in a lake are caught, tagged and put back.
Later, a second sample of 100 fish is caught. 8 of these are tagged.
(a) Work out an estimate for the number of fish in a lake.
(b) State one assumption you have made.

(a) The proportion tagged in the second sample is 8 out of 100

This should match the proportion tagged in the whole population: $\frac{8}{100} = \frac{20}{N}$

N = 20 × 100 ÷ 8 = 250

(b) No fish are born, die or leave between the samples, and the fish mix evenly

Answer: (a) 250 (b) The population stays the same between samples

2 Factorising quadratics

Factorise $8x^2 + 10x + 3$

8 × 3 = 24. Two numbers that multiply to 24 and add to 10 are 4 and 6

Split the middle term: $8x^2 + 4x + 6x + 3$
Factorise in pairs: $4x(2x + 1) + 3(2x + 1)$
$= (4x + 3)(2x + 1)$
Answer: (4x + 3)(2x + 1)

3 Cyclic quadrilaterals

A, B, C and D are points on a circle. Angle DAB = (3x + 20)°, angle BCD = (2x + 25)° and angle ABC = (2x + 5)°.
(a) Work out the value of x.
(b) Work out the size of angle ADC.
The diagram is not drawn accurately.

(a) Opposite angles of a cyclic quadrilateral add up to 180°
(3x + 20) + (2x + 25) = 180
5x + 45 = 180, so 5x = 135
x = 27
(b) Angle ABC = 2 × 27 + 5 = 59°
Angle ADC = 180 − 59 = 121°
Answer: (a) x = 27 (b) 121°

4 Direct proportion formulae

L is directly proportional to the cube of t.
When t = 5, L = 375.
(a) Find a formula for L in terms of t.
(b) Work out the positive value of t when L = 24.

(a) $L = kt^3$
375 = k × 125, so k = 375 ÷ 125 = 3
$L = 3t^3$
(b) $24 = 3t^3$, so $t^3 = 24 \div 3 = 8$, giving t = 2
Answer: (a) $L = 3t^3$ (b) t = 2

5 Simple and compound interest

Nia invests £3,000 for 4 years at 2% per year simple interest.
Kofi invests £3,000 for 4 years at 2% per year compound interest.
Work out how much more interest Kofi earns than Nia.
Give your answer to the nearest penny. (Calculator)

Simple interest: £3,000 × 2% × 4 = £240
Compound: £3,000 × 1.02^4 = £3,247.30
Compound interest = £3,247.30 − £3,000 = £247.30
Difference = £247.30 − £240 = £7.30
Answer: £7.30

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6 Pythagoras and quadratic equations

The lengths of the sides of a right-angled triangle are $(x - 8)$ cm, $(x - 9)$ cm and $(x - 7)$ cm.
The longest side is $(x - 7)$ cm.
(a) Show that $x^2 - 20x + 96 = 0$
(b) Hence work out the area of the triangle.

(a) Pythagoras: $(x - 8)^2 + (x - 9)^2 = (x - 7)^2$
Expand: $x^2 - 16x + 64 + x^2 - 18x + 81 = x^2 - 14x + 49$
Rearrange: $x^2 - 20x + 96 = 0$
(b) Factorise: $(x - 12)(x - 8) = 0$, so $x = 12$ (the other root, $x = 8$, does not give valid side lengths)
The shorter sides are 4 cm and 3 cm, so area $= \frac{1}{2} \times 4 \times 3 = 6$ cm^2

Answer: (b) 6 cm^2

7 Exact trigonometric ratios from a given ratio

x is an acute angle and $\cos x = \frac{1}{13}$.
(a) Find the exact value of sin x.
(b) Find the exact value of tan x. Give each answer as a fraction or a surd. Do not use a calculator.

(a) $\sin^2 x + \cos^2 x = 1$, so $\sin^2 x = 1 - (\frac{1}{13})^2 = \frac{168}{169}$
x is acute, so sin x is positive: $\sin x = \frac{2\sqrt{42}}{13}$
(b) $\tan x = \frac{\sin x}{\cos x} = \frac{2\sqrt{42}}{13} \div \frac{1}{13} = 2\sqrt{42}$

Answer: (a) $\frac{2\sqrt{42}}{13}$ (b) $2\sqrt{42}$

8 Mixtures and percentage profit

A shop mixes two types of tea.
Type A costs £7.00 per kg and type B costs £4.00 per kg.
They are mixed in the ratio A : B = 4 : 1.
The mixture is sold for £8.96 per kg.
Work out the percentage profit.

Cost of 5 kg of mixture = 4 × £7.00 + 1 × £4.00 = £32.00
Cost per kg = £32.00 ÷ 5 = £6.40
Profit per kg = £8.96 − £6.40 = £2.56
Percentage profit = 2.56 ÷ 6.40 × 100 = 40%
Answer: 40%

9 Counting factors

$N = 2^5 \times 3 \times 5$
How many of the factors of N are even numbers?

An even factor must contain at least one 2
Power of 2: 1 to 5, that is 5 choices
Power of 3: 0 to 1, that is 2 choices
Power of 5: 0 to 1, that is 2 choices
Total = 5 × 2 × 2 = 20
Answer: 20

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10 Iteration: rearranging equations

(a) Show that the equation $x^2 - x - 4 = 0$ can be rearranged to give $x = 1 + \frac{4}{x}$ (Calculator)

(b) Use the iteration formula $x_{n+1} = 1 + \frac{4}{x_n}$ with $x_0 = 2$ to work out x_1 and x_2. Give each answer correct to 3 decimal places.

(a) Rearrange $x^2 - x - 4 = 0$: $x^2 - x = 4$

Factorise: $x(x - 1) = 4$

Divide by x: $x - 1 = \frac{4}{x}$, so $x = 1 + \frac{4}{x}$

(b) $x_1 = 1 + \frac{4}{2} = 3.00000...$

$x_2 = 1 + \frac{4}{x_1} = 2.33333...$

$x_1 = 3.000$ and $x_2 = 2.333$ (3 d.p.)

Answer: (b) $x_1 = 3.000$, $x_2 = 2.333$