Day 1 **S Tier 10-a-day: Model Solutions**

1 Rationalising the denominator

Show that $\frac{6}{(3-\sqrt{3})}$ can be written as $3+\sqrt{3}$

Multiply top and bottom by the conjugate $(3+\sqrt{3})$:

$\frac{6(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$

Denominator: $9-3=6$

$\frac{6(3+\sqrt{3})}{6} = 3+\sqrt{3}$

Answer: $3+\sqrt{3}$

2 Ratio and algebra

The ratio of a to b is 2 : 3.
When 12 is added to each of a and b, the ratio becomes 3 : 4.
Work out the values of a and b.

Let $a = 2k$ and $b = 3k$.

$\frac{2k+12}{3k+12} = \frac{3}{4}$, so $4(2k+12) = 3(3k+12)$

$8k + 48 = 9k + 36$, so $k = 12$

Answer: a = 24, b = 36

3 Composite functions

$f(x) = 2x + 3$ and $g(x) = x^2 - 1$

(a) Find gf(x). Give your answer in the form $ax^2 + bx + c$.
(b) Solve $gf(x) = 24$

(a) $gf(x) = g(2x+3) = (2x+3)^2 - 1 = 4x^2 + 12x + 9 - 1 = 4x^2 + 12x + 8$

(b) $4x^2 + 12x + 8 = 24$, so $4x^2 + 12x - 16 = 0$

$x^2 + 3x - 4 = 0$, so $(x+4)(x-1) = 0$

Answer: (a) $4x^2 + 12x + 8$ (b) x = 1 or x = −4

4 Histograms

The histogram shows the times taken by some people to complete a task.
(a) Work out the number of people with a time in the class $20 < t \leq 35$.
(b) Estimate the number of people with a time greater than 30 minutes.

Frequency = frequency density × class width.

(a) $1.6 \times 15 = 24$

(b) From 30 to 35: $1.6 \times 5 = 8$. From 35 to 60: $0.4 \times 25 = 10$.

$8 + 10 = 18$

Answer: (a) 24 (b) 18

5 Circle theorems

PA and PB are tangents to a circle, centre O.
C is a point on the circle.
Angle APB = 40°.
(a) Work out the size of angle AOB.
(b) Work out the size of angle ACB.
The diagram is not drawn accurately.

(a) Angles OAP and OBP are 90° (a tangent meets the radius at 90°).
Angles in the quadrilateral OAPB: $360 - 90 - 90 - 40 = 140°$

(b) The angle at the centre is twice the angle at the circumference: $140 \div 2 = 70°$

Answer: (a) 140° (b) 70°

Day 1 **S Tier 10-a-day: Model Solutions**

6 Iteration

(a) Show that the equation $x^3 - 3x - 5 = 0$ has a solution between x = 2 and x = 3.
(b) Using $x_{n+1} = \sqrt[3]{3x_n + 5}$ with $x_0 = 2$, work out x_1 and x_2.
Give your answers correct to 3 decimal places. (Calculator)

(a) Let $f(x) = x^3 - 3x - 5$. $f(2) = 8 - 6 - 5 = -3$ and $f(3) = 27 - 9 - 5 = 13$.
There is a change of sign and f is continuous, so there is a solution between 2 and 3.
(b) $x_1 = \sqrt[3]{3 \times 2 + 5} = \sqrt[3]{11} = 2.2239...$
$x_2 = \sqrt[3]{3 \times 2.2239... + 5} = 2.2684...$
Answer: (b) $x_1 = 2.224$, $x_2 = 2.268$

7 Algebraic proof

Prove that the sum of the squares of any two consecutive odd numbers is always 2 more than a multiple of 8.

Let the odd numbers be $2n - 1$ and $2n + 1$, where n is an integer.
$(2n - 1)^2 + (2n + 1)^2 = 4n^2 - 4n + 1 + 4n^2 + 4n + 1$
$= 8n^2 + 2$
$8n^2$ is a multiple of 8, so the sum is 2 more than a multiple of 8.
Answer: $8n^2 + 2$, which is 2 more than a multiple of 8

8 Vectors

OAB is a triangle. OA = a and OB = b.
M is the midpoint of OA.
N is the point on AB such that AN : NB = 1 : 2.
(a) Find MN in terms of a and b.
(b) The line MN is extended to meet OB extended at the point P. Find OP in terms of b.

(a) AB = b − a, so AN = $\frac{1}{3}(b - a)$

ON = $a + \frac{1}{3}(b - a) = \frac{2}{3}a + \frac{1}{3}b$, and OM = $\frac{1}{2}a$

MN = ON − OM = $\frac{1}{6}a + \frac{1}{3}b = \frac{1}{6}(a + 2b)$

(b) P is on line MN, so MP = $\lambda(a + 2b)$ and OP = $\frac{1}{2}a + \lambda(a + 2b)$

OP is a multiple of b, so the a terms cancel: $\frac{1}{2} + \lambda = 0$, so $\lambda = -\frac{1}{2}$

OP = $2\lambda b = -b$

Answer: (a) $\frac{1}{6}a + \frac{1}{3}b$ (b) OP = $-b$

9 Conditional probability

A bag contains 4 red counters and n blue counters.
Two counters are taken at random, without replacement.
The probability that both counters are red is $\frac{2}{15}$.

(a) Show that $n^2 + 7n - 78 = 0$
(b) Hence find the value of n.

(a) There are n + 4 counters. P(both red) = $\frac{4}{(n + 4)} \times \frac{3}{(n + 3)} = \frac{12}{(n + 4)(n + 3)}$

$\frac{12}{(n + 4)(n + 3)} = \frac{2}{15}$, so $(n + 4)(n + 3) = 90$

$n^2 + 7n + 12 = 90$, so $n^2 + 7n - 78 = 0$
(b) $(n + 13)(n - 6) = 0$, so $n = -13$ or $n = 6$. A number of counters cannot be negative, so n = 6.
Answer: (b) n = 6

Day 1 **S Tier 10-a-day: Model Solutions**

10 Equation of a tangent to a circle

A circle has equation $x^2 + y^2 = 25$.
The point P(3, 4) lies on the circle.
Find the equation of the tangent to the circle at P.
Give your answer in the form ax + by = c.

The radius to P has gradient $\frac{4}{3}$.

The tangent is perpendicular to the radius, so its gradient is $-\frac{3}{4}$.

$y - 4 = -\frac{3}{4}(x - 3)$

Multiply by 4: $4y - 16 = -3x + 9$

Answer: 3x + 4y = 25