Day 93 **IGCSE 10-a-day: Model Solutions**

1 Square roots as powers

$y = -4x^{3/2} - 8\sqrt{x}$

Find $\frac{dy}{dx}$. Give each term of your answer in the form ax^n.

Write as powers of x: $y = -4x^{3/2} - 8x^{1/2}$

$-4x^{3/2} \to -6x^{1/2}$, $-8x^{1/2} \to -4x^{-1/2}$

$\frac{dy}{dx} = -6x^{1/2} - 4x^{-1/2}$

Answer: $\mathbf{\frac{dy}{dx} = -6x^{1/2} - 4x^{-1/2}}$

2 Factorising fully

Factorise fully $x^4 - 256$

$x^4 - 256 = (x^2)^2 - 16^2 = (x^2 - 16)(x^2 + 16)$

$x^2 - 16$ is also a difference of two squares: $(x - 4)(x + 4)$

Answer: $\mathbf{(x^2 + 16)(x - 4)(x + 4)}$

3 Fraction problems

A bag contains counters.

$\frac{2}{3}$ of the counters are red and $\frac{1}{5}$ of the counters are blue.

The rest of the counters, 6 of them, are green.

How many counters are in the bag?

Fraction that are green $= 1 - \frac{2}{3} - \frac{1}{5} = \frac{2}{15}$

$\frac{2}{15}$ of the total is 6

Total $= 6 \div \frac{2}{15} = 45$

Answer: 45

4 Stationary points of a quartic

A curve has equation $y = x^4 - 2x^2 + 3$.

Find the coordinates of the stationary points of the curve and determine the nature of each.

$\frac{dy}{dx} = 4x^3 - 4x = 4x(x - 1)(x + 1)$

$\frac{dy}{dx} = 0$ gives $x = 0$, $x = 1$ or $x = -1$

$y = 3$ when $x = 0$, and $y = 2$ when $x = \pm 1$

$\frac{d^2y}{dx^2} = 12x^2 - 4$: it is -4 at $x = 0$ (maximum) and 8 at $x = \pm 1$ (minimum)

Answer: (0, 3) maximum; (1, 2) and (−1, 2) minima

5 Counting subsets

The set A has 6 elements.

(a) Write down the number of subsets of A.

(b) Write down the number of proper subsets of A.

(a) Each element is either in or out of a subset: $2^6 = 64$

(b) Remove the set A itself: $64 - 1 = 63$

Answer: (a) 64 (b) 63

6 Fractions removed from a ratio

The ratio of men to women people at a meeting is 5 : 3.

Then $\frac{1}{4}$ of the men and $\frac{1}{2}$ of the women are taken out.

Work out the ratio of men to women now.

Left: $\frac{3}{4} \times 5 = \frac{15}{4}$ of the men part, $\frac{1}{2} \times 3 = \frac{3}{2}$ of the women part

$\frac{15}{4} : \frac{3}{2}$

Multiply by 4: 15 : 6 = 5 : 2

Answer: 5 : 2

7 Fencing with a partition

A rectangular enclosure is x metres wide and y metres long. It is fenced all round and divided into three equal parts by two fences each of length x metres, parallel to the width.

The total length of fencing used is 96 m.

(a) Show that the area of the enclosure is $A = -2x^2 + 48x$.

(b) Find the maximum possible area, justifying that it is a maximum.

Fencing: $4x + 2y = 96$, so $y = \frac{96 - 4x}{2}$

$A = xy = \frac{x(96 - 4x)}{2} = -2x^2 + 48x$

$\frac{dA}{dx} = -4x + 48 = 0$ gives $x = 12$

$\frac{d^2A}{dx^2} = -4 < 0$, so a maximum

$A = 288$

Answer: 288 m²

8 A squared in terms of A and I

$A = \begin{pmatrix} 6 & -4 \\ 6 & 3 \end{pmatrix}$.

$A^2 = pA + qI$, where I is the 2 × 2 identity matrix.

Find the value of p and the value of q.

$A^2 = \begin{pmatrix} 12 & -36 \\ 54 & -15 \end{pmatrix}$

Top right entries: $-36 = p \times (-4)$, so $p = 9$

Top left entries: $12 = 9 \times 6 + q$, so $q = -42$

Check bottom right: $9 \times 3 - 42 = -15$

Answer: p = 9, q = −42

9 Cone made from a sector

A sector of a circle has radius 24 cm and angle 120°.

The sector is made into a cone by joining the two straight edges together.

(a) Work out the radius of the base of the cone.

(b) Work out the exact volume of the cone. Give your answer in terms of π, as a surd if necessary.

(a) Arc length of the sector = $\frac{120}{360} \times 2\pi \times 24 = 16\pi$ cm

This is the circumference of the base: $2\pi r = 16\pi$, so $r = 8$ cm

(b) Slant height = 24 cm, so height$^2 = 24^2 - 8^2 = 512$, height = $16\sqrt{2}$ cm

Volume = $\frac{1}{3} \times \pi \times 8^2 \times 16\sqrt{2} = \frac{1024\sqrt{2}}{3}\pi$ cm^3

Answer: (a) 8 cm (b) $\frac{1024\sqrt{2}}{3}\pi$ cm^3

10 Algebraic proof (harder)

Prove algebraically that $m^3 + 29m$ is a multiple of 6 for all integer values of m.

$m^3 + 29m = (m^3 - m) + 30m = (m - 1)m(m + 1) + 30m$
$(m - 1)m(m + 1)$ is the product of three consecutive integers, so it contains a multiple of 2 and a multiple of 3, so it is a multiple of 6
$30m = 6 \times 5m$ is also a multiple of 6
The sum of two multiples of 6 is a multiple of 6
Answer: Proved