Day 86 **IGCSE 10-a-day: Model Solutions**

1 Tangents parallel to a line

A curve has equation $y = x^3 - 3x^2 - x - 1$.
Find the equations of the two tangents to the curve that are parallel to the line $y = 8x + 6$.

Parallel lines have equal gradients: gradient = 8

$\frac{dy}{dx} = 3x^2 - 6x - 1 = 8$, so $3x^2 - 6x - 9 = 0$

$3(x + 1)(x - 3) = 0$ gives $x = -1$ or $x = 3$

At $x = -1$: $y = -4$, tangent $y - (-4) = 8(x - (-1))$

At $x = 3$: $y = -4$, tangent $y - (-4) = 8(x - 3)$

Answer: $y = 8x + 4$ and $y = 8x - 28$

2 Adding and scaling matrices

$A = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

(a) Work out A + B.
(b) Work out 2A − 2B.

(a) $A + B = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(b) $2A = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$ and $2B = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

$2A - 2B = \begin{pmatrix} 12 \\ 14 \end{pmatrix}$

Answer: (a) $\begin{pmatrix} \mathbf{6} \\ \mathbf{3} \end{pmatrix}$ (b) $\begin{pmatrix} \mathbf{12} \\ \mathbf{14} \end{pmatrix}$

3 Volume and surface area of composite solids

A solid cylinder of radius 6 cm and height 13 cm has a hemispherical hollow of the same radius cut out of the top.
Work out the volume of the solid that is left. Give your answer in terms of π.

Cylinder: $\pi \times 6^2 \times 13 = 468\pi$

Hemisphere: $\frac{2}{3} \times \pi \times 6^3 = 144\pi$

Volume left = $468\pi - 144\pi = 324\pi$ cm^3

Answer: 324π cm^3

4 Parallel vectors

A = (2, −2), B = (5, 5), C = (6, 1) and D = (k, 15).
AB is parallel to **CD**.
Find the value of k.

$\mathbf{AB} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} k - 6 \\ 14 \end{pmatrix}$

Vertical parts: $14 = \lambda \times 7$, so $\lambda = 2$

Horizontal: $k - 6 = 2 \times 3 = 6$

$k = 12$

Answer: $k = 12$

5 Fraction problems

A fuel tank is $\frac{3}{5}$ full.

48 more litres of petrol are added and the fuel tank is then $\frac{9}{10}$ full.

Work out the capacity of the fuel tank.

Fraction added = $\frac{9}{10} - \frac{3}{5} = \frac{3}{10}$

$\frac{3}{10}$ of the capacity is 48 litres

Capacity = $48 \div \frac{3}{10} = 160$

Answer: 160 litres

6 Transformations of graphs

The graph of y = f(x) has a minimum point at (−1, −4).
Write down the coordinates of the turning point of the graph of
(a) y = f(x) − 2
(b) y = f(2x) + 2

(a) Start with the turning point (−1, −4)
Apply the transformation to the point: (−1, −6)

(b) f(2x) changes the x-coordinate to $-\frac{1}{2}$, and f(x) + 2 changes the y-coordinate to −2

Turning point: $(-\frac{1}{2}, -2)$

Answer: (a) (−1, −6) (b) $(-\frac{1}{2}, -2)$

7 Mixtures and percentage profit

A shop mixes two types of flour.
Type A costs £11.00 per kg and type B costs £5.00 per kg.
They are mixed in the ratio A : B = 3 : 2.
The mixture is sold for £10.75 per kg.
Work out the percentage profit.

Cost of 5 kg of mixture = 3 × £11.00 + 2 × £5.00 = £43.00
Cost per kg = £43.00 ÷ 5 = £8.60
Profit per kg = £10.75 − £8.60 = £2.15
Percentage profit = 2.15 ÷ 8.60 × 100 = 25%
Answer: 25%

8 Turning points of a cubic

The diagram shows a sketch of the curve $y = 2x^3 + 6x^2 - 18x - 8$.
The curve has turning points at A and B.
Find the coordinates of A and B and state whether each is a maximum or a minimum.

$\frac{dy}{dx} = 6x^2 + 12x - 18 = 6(x + 3)(x - 1)$

$\frac{dy}{dx} = 0$ gives x = −3 or x = 1

y = 46 when x = −3 and y = −18 when x = 1

$\frac{d^2y}{dx^2} = 12x + 12$: at x = −3 it is −24 (maximum), at x = 1 it is 24 (minimum)

Answer: A (−3, 46) maximum, B (1, −18) minimum

9 Algebraic proof (harder)

Prove that $n(n + 1)(n + 2)(n + 3) + 1$ is a square number for all positive integer values of n.

$n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$

Let $m = n^2 + 3n$. Then the expression is $m(m + 2) + 1$

$= m^2 + 2m + 1 = (m + 1)^2$

$= (n^2 + 3n + 1)^2$, which is the square of an integer

Answer: Proved

10 Cylinder of fixed surface area

A closed cylinder (a container) has radius r cm, height h cm and total surface area 384π cm^2.

(a) Show that its volume, V cm^3, is $V = 192\pi r - \pi r^3$.

(b) Find the maximum value of V in terms of π, and show that $h = 2r$ when V is a maximum.

$2\pi r^2 + 2\pi rh = 384\pi$, so $h = \frac{192 - r^2}{r}$

$V = \pi r^2 h = \pi r(192 - r^2) = 192\pi r - \pi r^3$

$\frac{dV}{dr} = 192\pi - 3\pi r^2 = 0$, so $r^2 = 64$ and $r = 8$

$\frac{d^2V}{dr^2} = -6\pi r < 0$, so a maximum. $h = \frac{192 - 64}{8} = 16 = 2r$

$V = \pi \times 8^2 \times 16 = 1024\pi$

Answer: 1024π cm^3