Day 64 **IGCSE 10-a-day: Model Solutions**

1 Show that a line is a tangent

The line $y = -7x + 3$ meets the curve $y = -x^2 - 3x - 1$.
Show that the line is a tangent to the curve and find the coordinates of the point where they touch.

Equate: $-x^2 - 3x - 1 = -7x + 3$, so $-x^2 + 4x - 4 = 0$
Discriminant: $4^2 - 4 \times (-1) \times (-4) = 16 - 16 = 0$
One repeated root, so the line touches the curve at one point only: it is a tangent
$x = \frac{-4}{2 \times (-1)} = 2$ and $y = -11$
Answer: (2, −11)

2 Find unknown constants in a function

$f(x) = ax + b$, where a and b are constants.
$f(4) = -2$ and $f(0) = -6$.
Find the value of a and the value of b.

$4a + b = -2$ and $0 + b = -6$
Subtract: $4a = 4$, so $a = 1$
$b = -2 - 1 \times 4 = -6$
Answer: $a = 1$, $b = -6$

3 Vectors: intersecting lines

OAB is a triangle. OA = $\mathbf{a}$ and OB = $\mathbf{b}$.
M is on OA with OM : MA = 2 : 1. N is on OB with ON : NB = 2 : 3. AN and BM intersect at X.
(a) Find BM in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Find OX in terms of $\mathbf{a}$ and $\mathbf{b}$.
The diagram is not drawn accurately.

(a) OM = $\frac{2}{3}\mathbf{a}$, so BM = $-\mathbf{b} + \frac{2}{3}\mathbf{a} = \frac{2}{3}\mathbf{a} - \mathbf{b}$

(b) X is on BM: BX = s BM, so OX = OB + s BM = $\frac{2}{3}s\mathbf{a} + (1 - s)\mathbf{b}$

X is on AN: OX = OA + t AN = OA + t(ON − OA) = $(1 - t)\mathbf{a} + \frac{2}{5}t\mathbf{b}$

Equate: $\frac{2}{3}s = 1 - t$ and $1 - s = \frac{2}{5}t$

Solving: $s = \frac{9}{11}$ and $t = \frac{5}{11}$

OX = $\frac{6}{11}\mathbf{a} + \frac{2}{11}\mathbf{b}$

Answer: (a) $\frac{2}{3}\mathbf{a} - \mathbf{b}$ (b) OX = $\frac{6}{11}\mathbf{a} + \frac{2}{11}\mathbf{b}$

4 Kinematics with roots

A particle moves in a straight line. After t seconds its displacement from O is s metres, where
$s = 8t^{\frac{3}{2}} + 2t^2$ $(t \geq 0)$
Find the velocity and the acceleration of the particle when $t = 4$.

$v = \frac{ds}{dt} = 4t + 12t^{\frac{1}{2}}$

$a = \frac{dv}{dt} = 4 + 6t^{-\frac{1}{2}}$

When $t = 4$: $v = 40$ and $a = 7$
Answer: $v = 40$ m/s, $a = 7$ m/s^2

5 Speed-time graphs

The graph shows the speed of a train during 19 seconds.
(a) Work out the acceleration during the first 6 seconds.
(b) Work out the average speed over the whole 19 seconds.

(a) Acceleration = change in speed ÷ time = $(20 - 8) \div 6 = 2$ m/s^2
(b) Distance = area under the graph
Trapezium: $\frac{8 + 20}{2} \times 6 = 84$; rectangle: $20 \times 9 = 180$; triangle: $\frac{1}{2} \times 4 \times 20 = 40$
Total = 304 m, so average speed = $304 \div 19 = 16$ m/s
Answer: (a) 2 m/s^2 (b) 16 m/s

6 Exponential graphs

The diagram shows the graph of $y = ka^x$, where k and a are constants and $a > 0$.
The curve passes through the points (1, 4) and (3, 64).
Find the value of k and the value of a.

Substitute the points: $k\,a^1 = 4$ and $k\,a^3 = 64$
Divide: $a^2 = \frac{64}{4} = 16$
$a = 4$ (positive, since $a > 0$)
From the first point: $k \times 4 = 4$, so $k = 1$
Answer: k = 1, a = 4

7 Fastest rate of growth

The number of fish in a lake, N, after t months is modelled by
$N = -t^3 + 9t^2 + 2t + 500$ for $0 \leq t \leq 12$
Find the time at which N is increasing most rapidly, and the rate of increase at this time.

$\frac{dN}{dt} = -3t^2 + 18t + 2$
Rate is greatest when $\frac{d^2N}{dt^2} = -6t + 18 = 0$, so $t = 3$
$\frac{d^2N}{dt^2}$ is decreasing (gradient -6), so the rate is a maximum
$\frac{dN}{dt} = 29$ when $t = 3$
Answer: t = 3, rate 29 per month

8 Successive percentage change

The population of a town is decreased by 25%.
The new amount is then decreased by 10%.
Work out the overall percentage change.

Multipliers: 0.75 and 0.9
$0.75 \times 0.9 = 0.675$
0.675 is a decrease of 32.5%
Answer: 32.5% decrease

9 Surds with unknowns

Given that $(a + \sqrt{11})^2 = 75 + 16\sqrt{11}$,
where a is a positive integer, find the value of a.

$(a + \sqrt{11})^2 = a^2 + 2a\sqrt{11} + 11$
Compare the $\sqrt{11}$ terms: $2a = 16$, so $a = 8$
Check the integer terms: $a^2 + 11 = 64 + 11 = 75$
Answer: a = 8

10 Intersecting lines in a triangle

$\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OP} = \frac{1}{2}\mathbf{a}$ and $\mathbf{OQ} = \frac{1}{4}\mathbf{b}$.

AQ and BP meet at X.
(a) Find $\mathbf{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Find the ratio AX : XQ.

Let $\mathbf{AX} = s\mathbf{AQ}$. Then $\mathbf{OX} = \mathbf{a} + s(-\mathbf{a} + \frac{1}{4}\mathbf{b}) = (1 - s)\mathbf{a} + \frac{1}{4}s\mathbf{b}$

Let $\mathbf{BX} = t\mathbf{BP}$. Then $\mathbf{OX} = \mathbf{b} + t(\frac{1}{2}\mathbf{a} - \mathbf{b}) = \frac{1}{2}t\mathbf{a} + (1 - t)\mathbf{b}$

Compare coefficients: $1 - s = \frac{1}{2}t$ and $\frac{1}{4}s = 1 - t$

Solving gives $s = \frac{4}{7}$ and $t = \frac{6}{7}$

(a) $\mathbf{OX} = \frac{3}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}$

(b) $AX = \frac{4}{7} \times AQ$, so $AX : XQ = \frac{4}{7} : \frac{3}{7} = 4 : 3$

Answer: (a) $\frac{3}{7}\mathbf{a} + \frac{1}{7}\mathbf{b}$ (b) 4 : 3