Day 59 **IGCSE 10-a-day: Model Solutions**

1 Self-inverse function

$f(x) = \frac{-5x - 7}{x - k}$, where k is a constant and $x \neq k$.

The function f is its own inverse, so $f^{-1}(x) = f(x)$.
Find the value of k.

Let $y = \frac{-5x - 7}{x - k}$, so $y(x - k) = -5x - 7$

$xy - ky = -5x - 7$, so $x(y + 5) = ky - 7$

$f^{-1}(x) = \frac{kx - 7}{x + 5}$

For $f^{-1}(x) = f(x)$ the denominators must match: $x + 5 = x - k$

Answer: k = −5

2 Stratified sampling

The table shows the number of members in each of four groups.
(a) A stratified sample of 26 is taken. Work out how many come from Group B.
(b) A different stratified sample has 9 from Group D. Work out the size of this sample.

Total = 70 + 40 + 120 + 30 = 260

(a) Group B: $\frac{40}{260} \times 26 = 4$

(b) Fraction sampled = $9 \div 30 = \frac{3}{10}$, so sample size = $\frac{3}{10} \times 260 = 78$

Answer: (a) 4 (b) 78

3 Fraction problems

A fuel tank is $\frac{1}{8}$ full.

20 more litres of petrol are added and the fuel tank is then $\frac{5}{8}$ full.

Work out the capacity of the fuel tank.

Fraction added = $\frac{5}{8} - \frac{1}{8} = \frac{1}{2}$

$\frac{1}{2}$ of the capacity is 20 litres

Capacity = $20 \div \frac{1}{2} = 40$

Answer: 40 litres

4 Rate of change of area or volume

The volume, V cm³, of a sphere of radius r cm is $V = \frac{4}{3}\pi r^3$.

(a) Find $\frac{dV}{dr}$ when r = 4.

(b) Find the value of r when $\frac{dV}{dr} = 100\pi$.

$\frac{dV}{dr} = 4\pi r^2$

When r = 4: $\frac{dV}{dr} = 64\pi$

$4\pi r^2 = 100\pi$, so $4r^2 = 100$ and r = 5 (lengths are positive)

Answer: (a) 64π (b) r = 5

5 Variation: missing values

T is directly proportional to m.
When m = 3, T = 30.
When m = 6, T = a.
When m = b, T = 100.
Work out the values of a and b.

T = km, k = 30 ÷ 3 = 10
a = 10 × 6 = 60
100 = 10b so b = 10, b = 10
Answer: a = 60, b = 10

6 Perpendicular lines

The diagram shows the straight line L, which passes through the points A and B.
The line M is perpendicular to L and passes through B.
Find the equation of M in the form y = mx + c.

Gradient of L = $\frac{7-0}{0-2} = -\frac{7}{2}$

Perpendicular gradient = $-1 \div (-\frac{7}{2}) = \frac{2}{7}$

M passes through B(2, 0): $0 = \frac{2}{7} \times 2 + c$, so $c = -\frac{4}{7}$

$y = \frac{2}{7}x - \frac{4}{7}$

Answer: $y = \frac{2}{7}x - \frac{4}{7}$

7 Three-part ratios

Kira, Maya and Tariq share some sweets.
Kira : Maya = 7 : 6
Maya : Tariq = 2 : 6
(a) Work out the ratio Kira : Maya : Tariq.
(b) They share 186 sweets altogether. How many sweets does Maya get?

(a) Make the Maya part the same: 7 : 6 = 7 : 6 and 2 : 6 = 6 : 18
Kira : Maya : Tariq = 7 : 6 : 18
(b) Total parts = 31, so 1 part = 186 ÷ 31 = 6
Maya gets 6 × 6 = 36 sweets
Answer: (a) 7 : 6 : 18 (b) 36 sweets

8 Tangent meets the curve again

A curve has equation $y = x^3 + 2x^2 - 2x - 5$.
The tangent to the curve at the point A, where $x = -2$, meets the curve again at the point B.
Find the coordinates of B.

$\frac{dy}{dx} = 3x^2 + 4x - 2$, so the gradient at A is 2, and A = (−2, −1)
Tangent: y = 2x + 3
Equate: $x^3 + 2x^2 - 4x - 8 = 0$, which has a repeated root $x = -2$
Sum of the roots = −2, so −2 + (−2) + x = −2 and x = 2
y = 7
Answer: B (2, 7)

9 Loci and regions

ABCD is a rectangle with AB = 8 cm and AD = 13 cm.
A point P inside the rectangle is such that AP ≤ 8 cm and P is nearer to A than to B.
Work out the area of the region in which P can lie. Give your answer in terms of π and surds.

Nearer to A than to B: P is on the A side of the perpendicular bisector of AB, a distance 4 cm from A
AP ≤ 8 cm: P is inside the quarter circle of radius 8 cm centred at A

The bisector meets the arc where $\cos\theta = \frac{4}{8}$, so $\theta = 60°$ at A. The region is a right-angled triangle plus a sector of angle 30°

Triangle: base 4 cm, height $8\sin 60° = 4\sqrt{3}$ cm, area $= \frac{1}{2} \times 4 \times 4\sqrt{3} = 8\sqrt{3}$

Sector of angle 30° : $\frac{30}{360} \times \pi \times 8^2 = \frac{16}{3}\pi$

Area $= 8\sqrt{3} + \frac{16}{3}\pi$ cm^2

Answer: $(8\sqrt{3} + \frac{16}{3}\pi)$ cm^2

10 Normal meets the curve again (quadratic)

The normal to the curve $y = x^2 + 3x - 6$ at the point P where $x = -1$ meets the curve again at Q.
Find the coordinates of Q.

$\frac{dy}{dx} = 2x + 3$: at P, gradient = 1 and $y = -8$
Normal: $y = -x - 9$
Intersection: $x^2 + 4x + 3 = 0$
One root is $x = -1$, and the sum of the roots is -4
So $x_Q = -4 - (-1) = -3$
$y = -6$
Answer: $(-3, -6)$