Day 53 **IGCSE 10-a-day: Model Solutions**

1 Fastest rate of growth

The number of bacteria in a culture, N, after t hours is modelled by

$N = -t^3 + 15t^2 + 9t + 1000$ for $0 \leq t \leq 20$

Find the time at which N is increasing most rapidly, and the rate of increase at this time.

$\frac{dN}{dt} = -3t^2 + 30t + 9$

Rate is greatest when $\frac{d^2N}{dt^2} = -6t + 30 = 0$, so $t = 5$

$\frac{d^2N}{dt^2}$ is decreasing (gradient -6), so the rate is a maximum

$\frac{dN}{dt} = 84$ when $t = 5$

Answer: t = 5, rate 84 per hour

2 Intersecting lines in a triangle

$\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OP} = \frac{2}{5}\mathbf{a}$ and $\mathbf{OQ} = \frac{1}{2}\mathbf{b}$.

AQ and BP meet at X.

(a) Find $\mathbf{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Find the ratio AX : XQ.

Let $\mathbf{AX} = s\mathbf{AQ}$. Then $\mathbf{OX} = \mathbf{a} + s(-\mathbf{a} + \frac{1}{2}\mathbf{b}) = (1 - s)\mathbf{a} + \frac{1}{2}s\mathbf{b}$

Let $\mathbf{BX} = t\mathbf{BP}$. Then $\mathbf{OX} = \mathbf{b} + t(\frac{2}{5}\mathbf{a} - \mathbf{b}) = \frac{2}{5}t\mathbf{a} + (1 - t)\mathbf{b}$

Compare coefficients: $1 - s = \frac{2}{5}t$ and $\frac{1}{2}s = 1 - t$

Solving gives $s = \frac{3}{4}$ and $t = \frac{5}{8}$

(a) $\mathbf{OX} = \frac{1}{4}\mathbf{a} + \frac{3}{8}\mathbf{b}$

(b) $AX = \frac{3}{4} \times AQ$, so $AX : XQ = \frac{3}{4} : \frac{1}{4} = 3 : 1$

Answer: (a) $\frac{1}{4}\mathbf{a} + \frac{3}{8}\mathbf{b}$ (b) 3 : 1

3 Tangents and circle theorems

PA and PB are tangents to a circle, centre O. C is a point on the circle. Angle ACB = 45°.

(a) Work out the size of angle AOB.

(b) Work out the size of angle APB.

The diagram is not drawn accurately.

(a) The angle at the centre is twice the angle at the circumference: $2 \times 45 = 90°$

(b) Angles OAP and OBP are 90° (a tangent meets a radius at 90°)

Angle APB = $360 - 90 - 90 - 90 = 90°$

Answer: (a) 90° (b) 90°

4 Exact lengths and areas

A square has sides of length $(7 + \sqrt{3})$ cm.

Work out the area of the square.

Give your answer in the form $a + b\sqrt{3}$

Area = $(7 + \sqrt{3})^2$

$= 49 + 14\sqrt{3} + 3$

$= 52 + 14\sqrt{3}$

Answer: $52 + 14\sqrt{3}$ cm^2

5 Tangent to a curve with negative or fractional powers

A curve has equation $y = -3x^2 - \frac{6}{x}$.

Find the equation of the tangent to the curve at the point where $x = -1$.

$\frac{dy}{dx} = -6x + 6x^{-2}$

At $x = -1$: $y = 3$ and gradient $= 12$

$y - 3 = 12(x + 1)$

$y = 12x + 15$

Answer: $y = 12x + 15$

6 Equations with indices

Solve $5^{2x-4} = \frac{1}{25}$

Write the right-hand side as a power of 5: $\frac{1}{25} = 5^{-2}$

Equate the powers: $2x - 4 = -2$

$2x = 2$

$x = 1$

Answer: $x = 1$

7 Cumulative frequency graphs

The cumulative frequency graph gives information about the marks of 100 students in a test.

(a) Find an estimate for the median and the interquartile range.

(b) Estimate how many of the 100 had a mark greater than 65.

(a) Median at cumulative frequency 50, so median = 50

Lower quartile at 25, so LQ = 40. Upper quartile at 75, so UQ = 60

Interquartile range = 60 − 40 = 20

(b) At 65, cumulative frequency = 85

Number greater = 100 − 85 = 15

Answer: (a) median 50, IQR 20 (b) 15

8 Area of a triangle given three sides

Work out the area of triangle ABC.

Give your answer correct to 3 significant figures. (Calculator)

$\cos A = \frac{7^2 + 12^2 - 12^2}{2 \times 7 \times 12} = \frac{7}{24}$

$A = 73.042°$

Area $= \frac{1}{2} \times 7 \times 12 \times \sin 73.042° = 40.2$

Answer: 40.2 cm^2

9 Minimum surface area of a tank

A storage unit is a cuboid with a square base of side x metres and height h metres, and has a closed top. Its volume is 512 m^3.

(a) Show that the total outside surface area, A m^2, is $A = 2x^2 + \frac{2048}{x}$.

(b) Find the minimum possible value of A, showing that it is a minimum.

$V = x^2h = 512$, so $h = \frac{512}{x^2}$

$A = 2x^2 + 4xh = 2x^2 + 4x \times \frac{512}{x^2} = 2x^2 + \frac{2048}{x}$

$\frac{dA}{dx} = 4x - 2048x^{-2} = 0$ gives $x^3 = 512$, so $x = 8$

$\frac{d^2A}{dx^2} = 4 + 4096x^{-3} = 12 > 0$, so a minimum

$A = 384$

Answer: 384 m^2

10 Tangent to a circle

A circle has equation $x^2 + y^2 = 841$.
The point P(20, −21) lies on the circle.
(a) Find the equation of the tangent to the circle at P. Give your answer in the form ax + by = c, where a, b and c are integers.
(b) The tangent meets the x-axis at the point T. Write down the coordinates of T.

(a) The gradient of the radius OP is $-\frac{21}{20}$

The tangent is perpendicular to the radius, so its gradient is $-1 \div (-\frac{21}{20}) = \frac{20}{21}$

$y - (-21) = \frac{20}{21}(x - 20)$

Multiply out and rearrange: $20x - 21y = 841$

(b) On the x-axis y = 0, so 20x = 841

$x = \frac{841}{20}$, so T is $(\frac{841}{20}, 0)$

Answer: (a) $20x - 21y = 841$ (b) T = $(\frac{841}{20}, 0)$