Day 362 **IGCSE 10-a-day: Model Solutions**

1 Minimum gradient of a curve

A curve has equation $y = 2x^3 + 12x^2 + 23x$.
Find the minimum value of the gradient of the curve, and the value of x at which it occurs.

$\frac{dy}{dx} = 6x^2 + 24x + 23$

The gradient is least when $\frac{d^2y}{dx^2} = 0$: $12x + 24 = 0$, so $x = -2$

$\frac{d^2y}{dx^2}$ increases through zero here (its derivative is $12 > 0$), so this is a minimum

$\frac{dy}{dx} = -1$ when $x = -2$

Answer: minimum gradient -1 when $x = -2$

2 Rational and irrational numbers

Here are six numbers.

$\frac{\sqrt{20}}{\sqrt{5}}$, $\sqrt{50}$, $\sqrt{\frac{9}{16}}$, $\pi \div 2$, $(1 + \sqrt{2})^2$, $(2 + \sqrt{5})(2 - \sqrt{5})$

Write down the rational numbers.

Simplify each number:

$\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{4} = 2$; $\sqrt{50} = 5\sqrt{2}$; $\sqrt{\frac{9}{16}} = \frac{3}{4}$

$\pi \div 2$ is irrational; $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$; $(2 + \sqrt{5})(2 - \sqrt{5}) = 4 - 5 = -1$

Answer: $\frac{\sqrt{20}}{\sqrt{5}}$, $\sqrt{\frac{9}{16}}$, $(2 + \sqrt{5})(2 - \sqrt{5})$

3 Depreciation

A van is bought for £12,000.
Its value falls by 20% each year.
Work out its value after 3 years. (Calculator)

Multiplier = $1 - 20 \div 100 = 0.8$

Value = 12000×0.8^3
= £6,144

Answer: £6,144

4 Averages with algebra

Here are five numbers: $x + 5$, $x + 4$, $x - 2$, $x + 6$, $3x - 1$.
The mean of the five numbers is 15.
(a) Work out the value of x.
(b) Work out the median of the five numbers.

(a) Total = $(x + 5) + (x + 4) + (x - 2) + (x + 6) + (3x - 1) = 15 \times 5 = 75$
$7x + 12 = 75$, so $7x = 63$ and $x = 9$
The numbers are 14, 13, 7, 15, 26
(b) In order: 7, 13, 14, 15, 26, so the median is 14

Answer: (a) $x = 9$ (b) 14

5 Algebraic fractions: adding and subtracting

Write $\frac{2}{x+2} + \frac{3}{x-5}$ as a single fraction.

Simplify the numerator.

The common denominator is $(x + 2)(x - 5)$

$$\frac{2}{x+2} + \frac{3}{x-5} = \frac{2(x-5) + 3(x+2)}{(x+2)(x-5)}$$

Expand the numerator: $5x - 4$

$$\frac{5x-4}{(x+2)(x-5)}$$

Answer: $\mathbf{\frac{5x-4}{(x+2)(x-5)}}$

6 Points with a given gradient on y = ax + b/x

A curve has equation $y = x + \frac{4}{x}$.

Find the coordinates of the points on the curve where the gradient is -3.

$$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

Set $\frac{dy}{dx} = -3$: $4 = \frac{4}{x^2}$

$x^2 = 1$, so $x = 1$ or $x = -1$

$y = 5$ when $x = 1$ and $y = -5$ when $x = -1$

Answer: (1, 5) and (−1, −5)

7 Laws of indices

Write $\frac{16^{n+2} \times 2^{n-3}}{8^n}$ in the form 2^{an+c}

where a and c are integers.

$16 = 2^4$, $2 = 2^1$, $8 = 2^3$

$16^{n+2} = 2^{4(n+2)}$, $2^{n-3} = 2^{n-3}$, $8^n = 2^{3n}$

Exponent: $4(n + 2) + (n - 3) - 3n = 2n + 5$

Answer: a = 2, c = 5

8 Self-inverse function

$f(x) = \frac{5x-1}{x-k}$, where k is a constant and $x \neq k$.

The function f is its own inverse, so $f^{-1}(x) = f(x)$.

Find the value of k.

Let $y = \frac{5x-1}{x-k}$, so $y(x - k) = 5x - 1$

$xy - ky = 5x - 1$, so $x(y - 5) = ky - 1$

$f^{-1}(x) = \frac{kx-1}{x-5}$

For $f^{-1}(x) = f(x)$ the denominators must match: $x - 5 = x - k$

Answer: k = 5

9 Loci and regions

ABCD is a rectangle with AB = 13 cm and AD = 16 cm.
A point P inside the rectangle is such that AP ≤ 12 cm and P is nearer to AB than to AD.
Work out the area of the region in which P can lie. Give your answer in terms of π.

AP ≤ 12 cm: P is inside the quarter circle of radius 12 cm centred at A (it fits inside the rectangle)
Nearer to AB than AD: P is on the AB side of the angle bisector of angle DAB, which is the line through A at 45° to AB
The bisector cuts the quarter circle into two equal halves

Area = $\frac{1}{2} \times \frac{1}{4} \times \pi \times 12^2 = 18\pi$ cm^2

Answer: 18π cm^2

10 Curve touching the x-axis

The curve $y = x^3 - 6x^2 + 9x + k$, where k is a constant, has the x-axis as a tangent.
Find the two possible values of k.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 0$ gives x = 1 or x = 3
The x-axis is a tangent when a turning point has y = 0
x = 1: 4 + k = 0, so k = −4
x = 3: 0 + k = 0, so k = 0
Answer: k = −4 or k = 0